Year 12
Maths A-level
Induction
work

Thank you for choosing to study Mathematics in the sixth form at Holmleigh Park High School.

Over the course, you will study topics in Pure Maths, Mechanics and Statistics.

The Mathematics Department is committed to ensuring that you make good progress throughout your A level course. In order that you make the best possible start, we have prepared this booklet. It is vital that you spend time working through the questions in this booklet over the summer as you need to have a good knowledge of these topics before you commence your course in September. You should have met all the topics before at GCSE.

Work through what you need to from each chapter, making sure that you understand the examples. Then tackle the exercise to ensure you understand the topic thoroughly. The answers are at the back of the booklet. You will need to be organised so keep your work in a folder & mark any queries to ask at the beginning of term.

In the first or second week of term you will take a test to check how well you understand these topics, so it is important that you have completed the booklet by then. Use this introduction to give you a good start to your Year 12 work that will help you to enjoy, and benefit from, the course. The more effort you put in, right from the start, the better you will do.

Contents

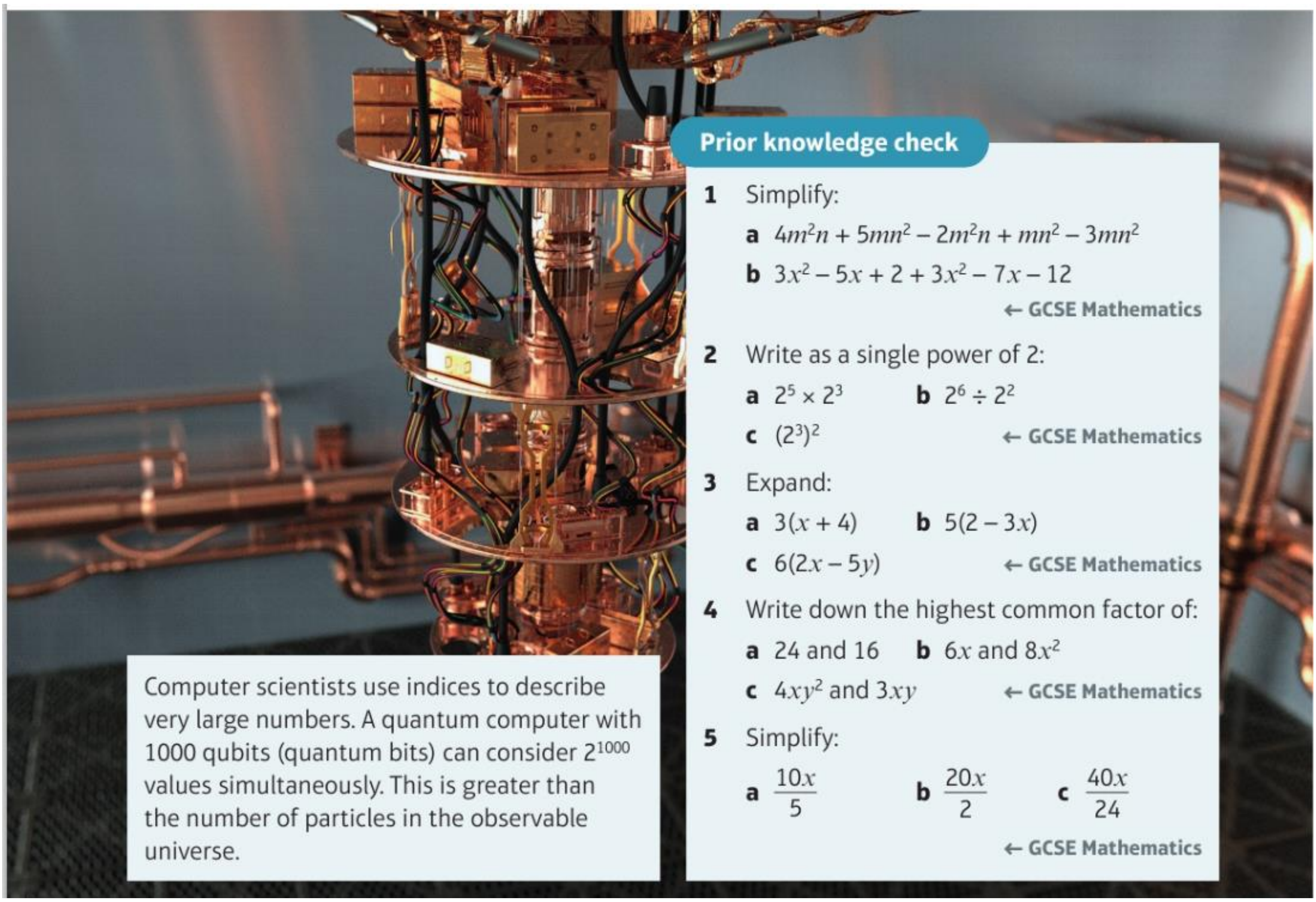
Algebraic expressions

1

Objectives

After completing this chapter you should be able to:

- Multiply and divide integer powers → **pages 2–3**
- Expand a single term over brackets and collect like terms → **pages 3–4**
- Expand the product of two or three expressions → **pages 4–6**
- Factorise linear, quadratic and simple cubic expressions → **pages 6–9**
- Know and use the laws of indices → **pages 9–11**
- Simplify and use the rules of surds → **pages 12–13**
- Rationalise denominators → **pages 13–16**



Computer scientists use indices to describe very large numbers. A quantum computer with 1000 qubits (quantum bits) can consider 2^{1000} values simultaneously. This is greater than the number of particles in the observable universe.

Prior knowledge check

1 Simplify:

a $4m^2n + 5mn^2 - 2m^2n + mn^2 - 3mn^2$

b $3x^2 - 5x + 2 + 3x^2 - 7x - 12$

← **GCSE Mathematics**

2 Write as a single power of 2:

a $2^5 \times 2^3$ **b** $2^6 \div 2^2$

c $(2^3)^2$ ← **GCSE Mathematics**

3 Expand:

a $3(x + 4)$ **b** $5(2 - 3x)$

c $6(2x - 5y)$ ← **GCSE Mathematics**

4 Write down the highest common factor of:

a 24 and 16 **b** $6x$ and $8x^2$

c $4xy^2$ and $3xy$ ← **GCSE Mathematics**

5 Simplify:

a $\frac{10x}{5}$ **b** $\frac{20x}{2}$ **c** $\frac{40x}{24}$

← **GCSE Mathematics**

1.1 Index laws

■ **You can use the laws of indices to simplify powers of the same base.**

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$

Notation

x^5 This is the **base**.

This is the **index**, **power** or **exponent**.

Example 1

Simplify these expressions:

a $x^2 \times x^5$ **b** $2r^2 \times 3r^3$ **c** $\frac{b^7}{b^4}$ **d** $6x^5 \div 3x^3$ **e** $(a^3)^2 \times 2a^2$ **f** $(3x^2)^3 \div x^4$

a $x^2 \times x^5 = x^{2+5} = x^7$

Use the rule $a^m \times a^n = a^{m+n}$ to simplify the index.

b $2r^2 \times 3r^3 = 2 \times 3 \times r^2 \times r^3$
$= 6 \times r^{2+3} = 6r^5$

Rewrite the expression with the numbers together and the r terms together.
$2 \times 3 = 6$
$r^2 \times r^3 = r^{2+3}$

c $\frac{b^7}{b^4} = b^{7-4} = b^3$

Use the rule $a^m \div a^n = a^{m-n}$ to simplify the index.

d $6x^5 \div 3x^3 = \frac{6}{3} \times \frac{x^5}{x^3}$
$= 2 \times x^2 = 2x^2$

$x^5 \div x^3 = x^{5-3} = x^2$

e $(a^3)^2 \times 2a^2 = a^6 \times 2a^2$
$= 2 \times a^6 \times a^2 = 2a^8$

Use the rule $(a^m)^n = a^{mn}$ to simplify the index.

$a^6 \times a^2 = a^{6+2} = a^8$

f $\frac{(3x^2)^3}{x^4} = 3^3 \times \frac{(x^2)^3}{x^4}$
$= 27 \times \frac{x^6}{x^4} = 27x^2$

Use the rule $(ab)^n = a^n b^n$ to simplify the numerator.
$(x^2)^3 = x^{2 \times 3} = x^6$

$\frac{x^6}{x^4} = x^{6-4} = x^2$

Example 2

Expand these expressions and simplify if possible:

a $-3x(7x - 4)$ **b** $y^2(3 - 2y^3)$

c $4x(3x - 2x^2 + 5x^3)$ **d** $2x(5x + 3) - 5(2x + 3)$

Watch out A minus sign outside brackets changes the sign of every term inside the brackets.

a $-3x(7x - 4) = -21x^2 + 12x$

$-3x \times 7x = -21x^{1+1} = -21x^2$
$-3x \times (-4) = +12x$

b $y^2(3 - 2y^3) = 3y^2 - 2y^5$

$y^2 \times (-2y^3) = -2y^{2+3} = -2y^5$

c $4x(3x - 2x^2 + 5x^3)$
$= 12x^2 - 8x^3 + 20x^4$

d $2x(5x + 3) - 5(2x + 3)$
$= 10x^2 + 6x - 10x - 15$
$= 10x^2 - 4x - 15$

Remember a minus sign outside the brackets changes the signs within the brackets.

Simplify $6x - 10x$ to give $-4x$.

Example 3

Simplify these expressions:

a $\dfrac{x^7 + x^4}{x^3}$ **b** $\dfrac{3x^2 - 6x^5}{2x}$ **c** $\dfrac{20x^7 + 15x^3}{5x^2}$

a $\dfrac{x^7 + x^4}{x^3} = \dfrac{x^7}{x^3} + \dfrac{x^4}{x^3}$

Divide each term of the numerator by x^3.

$= x^{7-3} + x^{4-3} = x^4 + x$

x^1 is the same as x.

b $\dfrac{3x^2 - 6x^5}{2x} = \dfrac{3x^2}{2x} - \dfrac{6x^5}{2x}$

Divide each term of the numerator by $2x$.

$= \dfrac{3}{2}x^{2-1} - 3x^{5-1} = \dfrac{3x}{2} - 3x^4$

Simplify each fraction:
$\dfrac{3x^2}{2x} = \dfrac{3}{2} \times \dfrac{x^2}{x} = \dfrac{3}{2} \times x^{2-1}$
$-\dfrac{6x^5}{2x} = -\dfrac{6}{2} \times \dfrac{x^5}{x} = -3 \times x^{5-1}$

c $\dfrac{20x^7 + 15x^3}{5x^2} = \dfrac{20x^7}{5x^2} + \dfrac{15x^3}{5x^2}$

Divide each term of the numerator by $5x^2$.

$= 4x^{7-2} + 3x^{3-2} = 4x^5 + 3x$

Exercise 1A

1 Simplify these expressions:

a $x^3 \times x^4$ **b** $2x^3 \times 3x^2$ **c** $\dfrac{k^3}{k^2}$

d $\dfrac{4p^3}{2p}$ **e** $\dfrac{3x^3}{3x^2}$ **f** $(y^2)^5$

g $10x^5 \div 2x^3$ **h** $(p^3)^2 \div p^4$ **i** $(2a^3)^2 \div 2a^3$

j $8p^4 \div 4p^3$ **k** $2a^4 \times 3a^5$ **l** $\dfrac{21a^3b^7}{7ab^4}$

m $9x^2 \times 3(x^2)^3$ **n** $3x^3 \times 2x^2 \times 4x^6$ **o** $7a^4 \times (3a^4)^2$

p $(4y^3)^3 \div 2y^3$ **q** $2a^3 \div 3a^2 \times 6a^5$ **r** $3a^4 \times 2a^5 \times a^3$

2 Expand and simplify if possible:

a $9(x-2)$ **b** $x(x+9)$ **c** $-3y(4-3y)$

d $x(y+5)$ **e** $-x(3x+5)$ **f** $-5x(4x+1)$

g $(4x+5)x$ **h** $-3y(5-2y^2)$ **i** $-2x(5x-4)$

j $(3x-5)x^2$ **k** $3(x+2)+(x-7)$ **l** $5x-6-(3x-2)$

m $4(c+3d^2)-3(2c+d^2)$ **n** $(r^2+3t^2+9)-(2r^2+3t^2-4)$

o $x(3x^2-2x+5)$ **p** $7y^2(2-5y+3y^2)$ **q** $-2y^2(5-7y+3y^2)$

r $7(x-2)+3(x+4)-6(x-2)$ **s** $5x-3(4-2x)+6$

t $3x^2-x(3-4x)+7$ **u** $4x(x+3)-2x(3x-7)$ **v** $3x^2(2x+1)-5x^2(3x-4)$

3 Simplify these fractions:

a $\dfrac{6x^4+10x^6}{2x}$ **b** $\dfrac{3x^5-x^7}{x}$ **c** $\dfrac{2x^4-4x^2}{4x}$

d $\dfrac{8x^3+5x}{2x}$ **e** $\dfrac{7x^7+5x^2}{5x}$ **f** $\dfrac{9x^5-5x^3}{3x}$

1.2 Expanding brackets

To find the **product** of two expressions you **multiply** each term in one expression by each term in the other expression.

$x\,\times$

$$(x+5)(4x-2y+3) = x(4x-2y+3)+5(4x-2y+3)$$
$$= 4x^2-2xy+3x+20x-10y+15$$
$$= 4x^2-2xy+23x-10y+15$$

$5\,\times$

Multiplying each of the 2 terms in the first expression by each of the 3 terms in the second expression gives $2 \times 3 = 6$ terms.

Simplify your answer by collecting like terms.

Example 4

Expand these expressions and simplify if possible:

a $(x+5)(x+2)$ **b** $(x-2y)(x^2+1)$ **c** $(x-y)^2$ **d** $(x+y)(3x-2y-4)$

a $(x+5)(x+2)$
$= x^2+2x+5x+10$
$= x^2+7x+10$

Multiply x by $(x+2)$ and then multiply 5 by $(x+2)$.

Simplify your answer by collecting like terms.

b $(x-2y)(x^2+1)$
$= x^3+x-2x^2y-2y$

$-2y \times x^2 = -2x^2y$

There are no like terms to collect.

c $(x-y)^2$
$= (x-y)(x-y)$
$= x^2 - xy - xy + y^2$
$= x^2 - 2xy + y^2$

$(x-y)^2$ means $(x-y)$ multiplied by itself.

$-xy - xy = -2xy$

d $(x+y)(3x-2y-4)$
$= x(3x-2y-4) + y(3x-2y-4)$
$= 3x^2 - 2xy - 4x + 3xy - 2y^2 - 4y$
$= 3x^2 + xy - 4x - 2y^2 - 4y$

Multiply x by $(3x-2y-4)$ and then multiply y by $(3x-2y-4)$.

Example 5

Expand these expressions and simplify if possible:

a $x(2x+3)(x-7)$ **b** $x(5x-3y)(2x-y+4)$ **c** $(x-4)(x+3)(x+1)$

a $x(2x+3)(x-7)$
$= (2x^2+3x)(x-7)$
$= 2x^3 - 14x^2 + 3x^2 - 21x$
$= 2x^3 - 11x^2 - 21x$

Start by expanding one pair of brackets: $x(2x+3) = 2x^2 + 3x$

You could also have expanded the second pair of brackets first: $(2x+3)(x-7) = 2x^2 - 11x - 21$
Then multiply by x.

b $x(5x-3y)(2x-y+4)$
$= (5x^2-3xy)(2x-y+4)$
$= 5x^2(2x-y+4) - 3xy(2x-y+4)$
$= 10x^3 - 5x^2y + 20x^2 - 6x^2y + 3xy^2 - 12xy$
$= 10x^3 - 11x^2y + 20x^2 + 3xy^2 - 12xy$

Be careful with minus signs. You need to change every sign in the second pair of brackets when you multiply it out.

c $(x-4)(x+3)(x+1)$
$= (x^2-x-12)(x+1)$
$= x^2(x+1) - x(x+1) - 12(x+1)$
$= x^3 + x^2 - x^2 - x - 12x - 12$
$= x^3 - 13x - 12$

Choose one pair of brackets to expand first, for example:
$(x-4)(x+3) = x^2 + 3x - 4x - 12$
$= x^2 - x - 12$

You multiplied together three linear terms, so the final answer contains an x^3 term.

Exercise 1B

1 Expand and simplify if possible:

a $(x+4)(x+7)$ **b** $(x-3)(x+2)$ **c** $(x-2)^2$
d $(x-y)(2x+3)$ **e** $(x+3y)(4x-y)$ **f** $(2x-4y)(3x+y)$
g $(2x-3)(x-4)$ **h** $(3x+2y)^2$ **i** $(2x+8y)(2x+3)$
j $(x+5)(2x+3y-5)$ **k** $(x-1)(3x-4y-5)$ **l** $(x-4y)(2x+y+5)$
m $(x+2y-1)(x+3)$ **n** $(2x+2y+3)(x+6)$ **o** $(4-y)(4y-x+3)$
p $(4y+5)(3x-y+2)$ **q** $(5y-2x+3)(x-4)$ **r** $(4y-x-2)(5-y)$

2 Expand and simplify if possible:

a $5(x+1)(x-4)$ **b** $7(x-2)(2x+5)$ **c** $3(x-3)(x-3)$

d $x(x-y)(x+y)$ **e** $x(2x+y)(3x+4)$ **f** $y(x-5)(x+1)$

g $y(3x-2y)(4x+2)$ **h** $y(7-x)(2x-5)$ **i** $x(2x+y)(5x-2)$

j $x(x+2)(x+3y-4)$ **k** $y(2x+y-1)(x+5)$ **l** $y(3x+2y-3)(2x+1)$

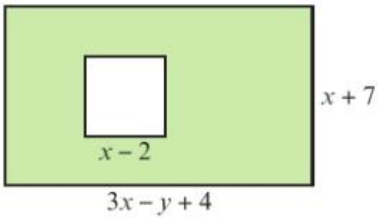
m $x(2x+3)(x+y-5)$ **n** $2x(3x-1)(4x-y-3)$ **o** $3x(x-2y)(2x+3y+5)$

p $(x+3)(x+2)(x+1)$ **q** $(x+2)(x-4)(x+3)$ **r** $(x+3)(x-1)(x-5)$

s $(x-5)(x-4)(x-3)$ **t** $(2x+1)(x-2)(x+1)$ **u** $(2x+3)(3x-1)(x+2)$

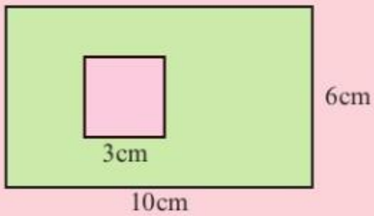
v $(3x-2)(2x+1)(3x-2)$ **w** $(x+y)(x-y)(x-1)$ **x** $(2x-3y)^3$

(P) **3** The diagram shows a rectangle with a square cut out.
The rectangle has length $3x - y + 4$ and width $x + 7$.
The square has length $x - 2$.
Find an expanded and simplified expression for the shaded area.



Problem-solving

Use the same strategy as you would use if the lengths were given as numbers:



(P) **4** A cuboid has dimensions $x + 2$ cm, $2x - 1$ cm and $2x + 3$ cm.
Show that the volume of the cuboid is $4x^3 + 12x^2 + 5x - 6$ cm^3.

(E/P) **5** Given that $(2x + 5y)(3x - y)(2x + y) = ax^3 + bx^2y + cxy^2 + dy^3$, where a, b, c and d are constants, find the values of a, b, c and d. **(2 marks)**

Challenge

Expand and simplify $(x + y)^4$.

Links You can use the binomial expansion to expand expressions like $(x + y)^4$ quickly. → **Section 8.3**

1.3 Factorising

You can write expressions as a **product of their factors**.

- **Factorising is the opposite of expanding brackets.**

Expanding brackets →

$$4x(2x + y) = 8x^2 + 4xy$$
$$(x + 5)^3 = x^3 + 15x^2 + 75x + 125$$
$$(x + 2y)(x - 5y) = x^2 - 3xy - 10y^2$$

← Factorising

Example 6

Factorise these expressions completely:

a $3x + 9$ **b** $x^2 - 5x$ **c** $8x^2 + 20x$ **d** $9x^2y + 15xy^2$ **e** $3x^2 - 9xy$

a $3x + 9 = 3(x + 3)$

3 is a common factor of $3x$ and 9.

b $x^2 - 5x = x(x - 5)$

x is a common factor of x^2 and $-5x$.

c $8x^2 + 20x = 4x(2x + 5)$

4 and x are common factors of $8x^2$ and $20x$. So take $4x$ outside the brackets.

d $9x^2y + 15xy^2 = 3xy(3x + 5y)$

3, x and y are common factors of $9x^2y$ and $15xy^2$. So take $3xy$ outside the brackets.

e $3x^2 - 9xy = 3x(x - 3y)$

x and $-3y$ have no common factors so this expression is completely factorised.

- **A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.**

Notation Real numbers are all the positive and negative numbers, or zero, including fractions and surds.

To factorise a quadratic expression:

- Find two factors of ac that add up to b
- Rewrite the b term as a sum of these two factors
- Factorise each pair of terms
- Take out the common factor

For the expression $2x^2 + 5x - 3$, $ac = -6 = -1 \times 6$ and $-1 + 6 = 5 = b$.

$2x^2 - x + 6x - 3$

$= x(2x - 1) + 3(2x - 1)$

$= (x + 3)(2x - 1)$

- $x^2 - y^2 = (x + y)(x - y)$

Notation An expression in the form $x^2 - y^2$ is called the **difference** of two squares.

Example 7

Factorise:

a $x^2 - 5x - 6$ **b** $x^2 + 6x + 8$ **c** $6x^2 - 11x - 10$ **d** $x^2 - 25$ **e** $4x^2 - 9y^2$

a $x^2 - 5x - 6$

$ac = -6$ and $b = -5$

So $x^2 - 5x - 6 = x^2 + x - 6x - 6$

$= x(x + 1) - 6(x + 1)$

$= (x + 1)(x - 6)$

Here $a = 1$, $b = -5$ and $c = -6$.

1. Work out the two factors of $ac = -6$ which add to give you $b = -5$. $-6 + 1 = -5$
2. Rewrite the b term using these two factors.
3. Factorise first two terms and last two terms.
4. $x + 1$ is a factor of both terms, so take that outside the brackets. This is now completely factorised.

b $x^2 + 6x + 8$

$= x^2 + 2x + 4x + 8$

$= x(x + 2) + 4(x + 2)$

$= (x + 2)(x + 4)$

$ac = 8$ and $2 + 4 = 6 = b$.

Factorise.

c $6x^2 - 11x - 10$

$= 6x^2 - 15x + 4x - 10$

$= 3x(2x - 5) + 2(2x - 5)$

$= (2x - 5)(3x + 2)$

$ac = -60$ and $4 - 15 = -11 = b$.

Factorise.

d $x^2 - 25$

$= x^2 - 5^2$

$= (x + 5)(x - 5)$

This is the difference of two squares as the two terms are x^2 and 5^2.

The two x terms, $5x$ and $-5x$, cancel each other out.

e $4x^2 - 9y^2$

$= 2^2x^2 - 3^2y^2$

$= (2x + 3y)(2x - 3y)$

This is the same as $(2x)^2 - (3y)^2$.

Example 8

Factorise completely:

a $x^3 - 2x^2$ **b** $x^3 - 25x$ **c** $x^3 + 3x^2 - 10x$

a $x^3 - 2x^2 = x^2(x - 2)$

You can't factorise this any further.

b $x^3 - 25x = x(x^2 - 25)$

$= x(x^2 - 5^2)$

$= x(x + 5)(x - 5)$

x is a common factor of x^3 and $-25x$. So take x outside the brackets.

$x^2 - 25$ is the difference of two squares.

c $x^3 + 3x^2 - 10x = x(x^2 + 3x - 10)$

$= x(x + 5)(x - 2)$

Write the expression as a product of x and a quadratic factor.

Factorise the quadratic to get three linear factors.

Exercise 1C

1 Factorise these expressions completely:

a $4x + 8$	**b** $6x - 24$	**c** $20x + 15$
d $2x^2 + 4$	**e** $4x^2 + 20$	**f** $6x^2 - 18x$
g $x^2 - 7x$	**h** $2x^2 + 4x$	**i** $3x^2 - x$
j $6x^2 - 2x$	**k** $10y^2 - 5y$	**l** $35x^2 - 28x$
m $x^2 + 2x$	**n** $3y^2 + 2y$	**o** $4x^2 + 12x$
p $5y^2 - 20y$	**q** $9xy^2 + 12x^2y$	**r** $6ab - 2ab^2$
s $5x^2 - 25xy$	**t** $12x^2y + 8xy^2$	**u** $15y - 20yz^2$
v $12x^2 - 30$	**w** $xy^2 - x^2y$	**x** $12y^2 - 4yx$

2 Factorise:

a $x^2 + 4x$ **b** $2x^2 + 6x$ **c** $x^2 + 11x + 24$
d $x^2 + 8x + 12$ **e** $x^2 + 3x - 40$ **f** $x^2 - 8x + 12$
g $x^2 + 5x + 6$ **h** $x^2 - 2x - 24$ **i** $x^2 - 3x - 10$
j $x^2 + x - 20$ **k** $2x^2 + 5x + 2$ **l** $3x^2 + 10x - 8$
m $5x^2 - 16x + 3$ **n** $6x^2 - 8x - 8$
o $2x^2 + 7x - 15$ **p** $2x^4 + 14x^2 + 24$
q $x^2 - 4$ **r** $x^2 - 49$
s $4x^2 - 25$ **t** $9x^2 - 25y^2$ **u** $36x^2 - 4$
v $2x^2 - 50$ **w** $6x^2 - 10x + 4$ **x** $15x^2 + 42x - 9$

Hint For part **n**, take 2 out as a common factor first. For part **p**, let $y = x^2$.

3 Factorise completely:

a $x^3 + 2x$ **b** $x^3 - x^2 + x$ **c** $x^3 - 5x$
d $x^3 - 9x$ **e** $x^3 - x^2 - 12x$ **f** $x^3 + 11x^2 + 30x$
g $x^3 - 7x^2 + 6x$ **h** $x^3 - 64x$ **i** $2x^3 - 5x^2 - 3x$
j $2x^3 + 13x^2 + 15x$ **k** $x^3 - 4x$ **l** $3x^3 + 27x^2 + 60x$

E/P **4** Factorise completely $x^4 - y^4$. **(2 marks)**

Problem-solving

Watch out for terms that can be written as a function of a function: $x^4 = (x^2)^2$

E **5** Factorise completely $6x^3 + 7x^2 - 5x$. **(2 marks)**

Challenge

Write $4x^4 - 13x^2 + 9$ as the product of four linear factors.

1.4 Negative and fractional indices

Indices can be negative numbers or fractions.

$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x,$

similarly $\underbrace{x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times \ldots \times x^{\frac{1}{n}}}_{n \text{ terms}} = x^{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}} = x^1 = x$

Notation **Rational** numbers are those that can be written as $\frac{a}{b}$ where a and b are integers.

■ **You can use the laws of indices with any rational power.**

- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$

Notation $a^{\frac{1}{2}} = \sqrt{a}$ is the positive square root of a. For example $9^{\frac{1}{2}} = \sqrt{9} = 3$ but $9^{\frac{1}{2}} \neq -3$.

Example 9

Simplify:

a $\dfrac{x^3}{x^{-3}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$ **c** $(x^3)^{\frac{2}{3}}$ **d** $2x^{1.5} \div 4x^{-0.25}$ **e** $\sqrt[3]{125x^6}$ **f** $\dfrac{2x^2 - x}{x^5}$

a $\dfrac{x^3}{x^{-3}} = x^{3-(-3)} = x^6$

Use the rule $a^m \div a^n = a^{m-n}$.

b $x^{\frac{1}{2}} \times x^{\frac{3}{2}} = x^{\frac{1}{2}+\frac{3}{2}} = x^2$

This could also be written as $\sqrt{x}$.
Use the rule $a^m \times a^n = a^{m+n}$.

c $(x^3)^{\frac{2}{3}} = x^{3 \times \frac{2}{3}} = x^2$

Use the rule $(a^m)^n = a^{mn}$.

d $2x^{1.5} \div 4x^{-0.25} = \frac{1}{2}x^{1.5-(-0.25)} = \frac{1}{2}x^{1.75}$

Use the rule $a^m \div a^n = a^{m-n}$.
$1.5 - (-0.25) = 1.75$

e $\sqrt[3]{125x^6} = (125x^6)^{\frac{1}{3}}$

$= (125)^{\frac{1}{3}}(x^6)^{\frac{1}{3}} = \sqrt[3]{125}(x^{6 \times \frac{1}{3}}) = 5x^2$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$.

f $\dfrac{2x^2 - x}{x^5} = \dfrac{2x^2}{x^5} - \dfrac{x}{x^5}$

$= 2 \times x^{2-5} - x^{1-5} = 2x^{-3} - x^{-4}$

Divide each term of the numerator by x^5.

$= \dfrac{2}{x^3} - \dfrac{1}{x^4}$

Using $a^{-m} = \dfrac{1}{a^m}$

Example 10

Evaluate:

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $25^{-\frac{3}{2}}$

a $9^{\frac{1}{2}} = \sqrt{9} = 3$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$. $9^{\frac{1}{2}} = \sqrt{9}$

b $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

This means the cube root of 64.

c $49^{\frac{3}{2}} = (\sqrt{49})^3$

$7^3 = 343$

Using $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.
This means the square root of 49, cubed.

d $25^{-\frac{3}{2}} = \dfrac{1}{25^{\frac{3}{2}}} = \dfrac{1}{(\sqrt{25})^3}$

$= \dfrac{1}{5^3} = \dfrac{1}{125}$

Using $a^{-m} = \dfrac{1}{a^m}$

Online Use your calculator to enter negative and fractional powers.

Example 11

Given that $y = \frac{1}{16}x^2$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{2}}$ **b** $4y^{-1}$

a $y^{\frac{1}{2}} = \left(\frac{1}{16}x^2\right)^{\frac{1}{2}}$

$= \frac{1}{\sqrt{16}}x^{2 \times \frac{1}{2}} = \frac{x}{4}$

b $4y^{-1} = 4\left(\frac{1}{16}x^2\right)^{-1}$

$= 4\left(\frac{1}{16}\right)^{-1}x^{2 \times (-1)} = 4 \times 16x^{-2}$

$= 64x^{-2}$

Substitute $y = \frac{1}{16}x^2$ into $y^{\frac{1}{2}}$.

$\left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{16}}$ and $(x^2)^{\frac{1}{2}} = x^{2 \times \frac{1}{2}}$

$\left(\frac{1}{16}\right)^{-1} = 16$ and $x^{2 \times -1} = x^{-2}$

Problem-solving

Check that your answers are in the correct form. If k and n are constants they could be positive or negative, and they could be integers, fractions or surds.

Exercise 1D

1 Simplify:

a $x^3 \div x^{-2}$ **b** $x^5 \div x^7$ **c** $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$

d $(x^2)^{\frac{3}{2}}$ **e** $(x^3)^{\frac{5}{3}}$ **f** $3x^{0.5} \times 4x^{-0.5}$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}}$ **h** $5x^{\frac{7}{5}} \div x^{\frac{2}{5}}$ **i** $3x^4 \times 2x^{-5}$

j $\sqrt{x} \times \sqrt[3]{x}$ **k** $(\sqrt{x})^3 \times (\sqrt[3]{x})^4$ **l** $\frac{(\sqrt[3]{x})^2}{\sqrt{x}}$

2 Evaluate:

a $25^{\frac{1}{2}}$ **b** $81^{\frac{3}{2}}$ **c** $27^{\frac{1}{3}}$

d 4^{-2} **e** $9^{-\frac{1}{2}}$ **f** $(-5)^{-3}$

g $\left(\frac{3}{4}\right)^0$ **h** $1296^{\frac{3}{4}}$ **i** $\left(\frac{25}{16}\right)^{\frac{3}{2}}$

j $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ **k** $\left(\frac{6}{5}\right)^{-1}$ **l** $\left(\frac{343}{512}\right)^{-\frac{2}{3}}$

3 Simplify:

a $(64x^{10})^{\frac{1}{2}}$ **b** $\frac{5x^3 - 2x^2}{x^5}$ **c** $(125x^{12})^{\frac{1}{3}}$ **d** $\frac{x + 4x^3}{x^3}$

e $\frac{2x + x^2}{x^4}$ **f** $\left(\frac{4}{9}x^4\right)^{\frac{3}{2}}$ **g** $\frac{9x^2 - 15x^5}{3x^3}$ **h** $\frac{5x + 3x^2}{15x^3}$

E **4** **a** Find the value of $81^{\frac{1}{4}}$. **(1 mark)**

b Simplify $x(2x^{-\frac{1}{3}})^4$. **(2 marks)**

E **5** Given that $y = \frac{1}{8}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(2 marks)**

b $\frac{1}{2}y^{-2}$ **(2 marks)**

1.5 Surds

If n is an integer that is **not** a square number, then any multiple of $\sqrt{n}$ is called a surd.
Examples of surds are $\sqrt{2}$, $\sqrt{19}$ and $5\sqrt{2}$.

Surds are examples of **irrational numbers**.
The decimal expansion of a surd is never-ending and never repeats, for example $\sqrt{2} = 1.414213562\ldots$

Notation Irrational numbers cannot be written in the form $\frac{a}{b}$ where a and b are integers.
Surds are examples of **irrational numbers**.

You can use surds to write exact answers to calculations.

- **You can manipulate surds using these rules:**
  - $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
  - $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example 12

Simplify:

a $\sqrt{12}$ **b** $\frac{\sqrt{20}}{2}$ **c** $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

a $\sqrt{12} = \sqrt{(4 \times 3)}$

$= \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Look for a factor of 12 that is a square number. Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. $\sqrt{4} = 2$

b $\frac{\sqrt{20}}{2} = \frac{\sqrt{4} \times \sqrt{5}}{2}$

$\sqrt{20} = \sqrt{4} \times \sqrt{5}$

$\sqrt{4} = 2$

$= \frac{2 \times \sqrt{5}}{2} = \sqrt{5}$

Cancel by 2.

c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

$= 5\sqrt{6} - 2\sqrt{6}\sqrt{4} + \sqrt{6} \times \sqrt{49}$

$\sqrt{6}$ is a common factor.

$= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49})$

Work out the square roots $\sqrt{4}$ and $\sqrt{49}$.

$= \sqrt{6}(5 - 2 \times 2 + 7)$

$5 - 4 + 7 = 8$

$= \sqrt{6}(8)$

$= 8\sqrt{6}$

Example 13

Expand and simplify if possible:

a $\sqrt{2}(5-\sqrt{3})$ **b** $(2-\sqrt{3})(5+\sqrt{3})$

a $\sqrt{2}(5-\sqrt{3})$

$= 5\sqrt{2} - \sqrt{2}\sqrt{3}$

$= 5\sqrt{2} - \sqrt{6}$

b $(2-\sqrt{3})(5+\sqrt{3})$

$= 2(5+\sqrt{3}) - \sqrt{3}(5+\sqrt{3})$

$= 10 + 2\sqrt{3} - 5\sqrt{3} - \sqrt{9}$

$= 7 - 3\sqrt{3}$

$\sqrt{2}\times 5 - \sqrt{2}\times\sqrt{3}$

Using $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$

Expand the brackets completely before you simplify.

Collect like terms: $2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$

Simplify any roots if possible: $\sqrt{9} = 3$

Exercise 1E

1 Do not use your calculator for this exercise. Simplify:

a $\sqrt{28}$ **b** $\sqrt{72}$ **c** $\sqrt{50}$

d $\sqrt{32}$ **e** $\sqrt{90}$ **f** $\dfrac{\sqrt{12}}{2}$

g $\dfrac{\sqrt{27}}{3}$ **h** $\sqrt{20}+\sqrt{80}$ **i** $\sqrt{200}+\sqrt{18}-\sqrt{72}$

j $\sqrt{175}+\sqrt{63}+2\sqrt{28}$ **k** $\sqrt{28}-2\sqrt{63}+\sqrt{7}$ **l** $\sqrt{80}-2\sqrt{20}+3\sqrt{45}$

m $3\sqrt{80}-2\sqrt{20}+5\sqrt{45}$ **n** $\dfrac{\sqrt{44}}{\sqrt{11}}$ **o** $\sqrt{12}+3\sqrt{48}+\sqrt{75}$

2 Expand and simplify if possible:

a $\sqrt{3}(2+\sqrt{3})$ **b** $\sqrt{5}(3-\sqrt{3})$ **c** $\sqrt{2}(4-\sqrt{5})$

d $(2-\sqrt{2})(3+\sqrt{5})$ **e** $(2-\sqrt{3})(3-\sqrt{7})$ **f** $(4+\sqrt{5})(2+\sqrt{5})$

g $(5-\sqrt{3})(1-\sqrt{3})$ **h** $(4+\sqrt{3})(2-\sqrt{3})$ **i** $(7-\sqrt{11})(2+\sqrt{11})$

E **3** Simplify $\sqrt{75}-\sqrt{12}$ giving your answer in the form $a\sqrt{3}$, where a is an integer. **(2 marks)**

1.6 Rationalising denominators

If a fraction has a surd in the denominator, it is sometimes useful to **rearrange** it so that the denominator is a **rational** number. This is called rationalising the denominator.

- **The rules to rationalise denominators are:**
  - **For fractions in the form $\dfrac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.**
  - **For fractions in the form $\dfrac{1}{a+\sqrt{b}}$, multiply the numerator and denominator by $a-\sqrt{b}$.**
  - **For fractions in the form $\dfrac{1}{a-\sqrt{b}}$, multiply the numerator and denominator by $a+\sqrt{b}$.**

Example 14

Rationalise the denominator of:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ **d** $\frac{1}{(1-\sqrt{3})^2}$

a $\frac{1}{\sqrt{3}} = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$ — Multiply the numerator and denominator by $\sqrt{3}$.

$= \frac{\sqrt{3}}{3}$ — $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

b $\frac{1}{3+\sqrt{2}} = \frac{1 \times (3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$ — Multiply numerator and denominator by $(3-\sqrt{2})$.

$= \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2}$ — $\sqrt{2} \times \sqrt{2} = 2$

$= \frac{3-\sqrt{2}}{7}$ — $9-2=7$, $-3\sqrt{2}+3\sqrt{2}=0$

c $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ — Multiply numerator and denominator by $\sqrt{5}+\sqrt{2}$.

$= \frac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$ — $-\sqrt{2}\sqrt{5}$ and $\sqrt{5}\sqrt{2}$ cancel each other out.

$= \frac{7+2\sqrt{10}}{3}$ — $\sqrt{5}\sqrt{2} = \sqrt{10}$

d $\frac{1}{(1-\sqrt{3})^2} = \frac{1}{(1-\sqrt{3})(1-\sqrt{3})}$ — Expand the brackets.

$= \frac{1}{1-\sqrt{3}-\sqrt{3}+\sqrt{9}}$ — Simplify and collect like terms. $\sqrt{9} = 3$

$= \frac{1}{4-2\sqrt{3}}$

$= \frac{1 \times (4+2\sqrt{3})}{(4-2\sqrt{3})(4+2\sqrt{3})}$ — Multiply the numerator and denominator by $4+2\sqrt{3}$.

$\sqrt{3} \times \sqrt{3} = 3$

$= \frac{4+2\sqrt{3}}{16+8\sqrt{3}-8\sqrt{3}-12}$ — $16-12=4$, $8\sqrt{3}-8\sqrt{3}=0$

$= \frac{4+2\sqrt{3}}{4} = \frac{2+\sqrt{3}}{2}$

Exercise 1F

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

E/P **4** Simplify $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ giving your answer in the form $p + q\sqrt{5}$, where p and q are rational numbers. **(4 marks)**

Problem-solving

You can check that your answer is in the correct form by writing down the values of p and q and checking that they are rational numbers.

Mixed exercise 1

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$ **c** $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand and simplify if possible:

a $(x + 3)(x - 5)$ **b** $(2x - 7)(3x + 1)$ **c** $(2x + 5)(3x - y + 2)$

3 Expand and simplify if possible:

a $x(x + 4)(x - 1)$ **b** $(x + 2)(x - 3)(x + 7)$ **c** $(2x + 3)(x - 2)(3x - 1)$

4 Expand the brackets:

a $3(5y + 4)$ **b** $5x^2(3 - 5x + 2x^2)$ **c** $5x(2x + 3) - 2x(1 - 3x)$ **d** $3x^2(1 + 3x) - 2x(3x - 2)$

5 Factorise these expressions completely:

a $3x^2 + 4x$ **b** $4y^2 + 10y$ **c** $x^2 + xy + xy^2$ **d** $8xy^2 + 10x^2y$

6 Factorise:

a $x^2 + 3x + 2$ **b** $3x^2 + 6x$ **c** $x^2 - 2x - 35$ **d** $2x^2 - x - 3$

e $5x^2 - 13x - 6$ **f** $6 - 5x - x^2$

7 Factorise:

a $2x^3 + 6x$ **b** $x^3 - 36x$ **c** $2x^3 + 7x^2 - 15x$

8 Simplify:

a $9x^3 \div 3x^{-3}$ **b** $(4^{\frac{3}{2}})^{\frac{1}{3}}$ **c** $3x^{-2} \times 2x^4$ **d** $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

9 Evaluate:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **b** $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

10 Simplify:

a $\frac{3}{\sqrt{63}}$ **b** $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

11 a Find the value of $35x^2 + 2x - 48$ when $x = 25$.

b By factorising the expression, show that your answer to part **a** can be written as the product of two prime factors.

12 Expand and simplify if possible:

a $\sqrt{2}(3 + \sqrt{5})$ **b** $(2 - \sqrt{5})(5 + \sqrt{3})$ **c** $(6 - \sqrt{2})(4 - \sqrt{7})$

13 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2} - 1}$ **c** $\frac{3}{\sqrt{3} - 2}$ **d** $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$ **e** $\frac{1}{(2 + \sqrt{3})^2}$ **f** $\frac{1}{(4 - \sqrt{7})^2}$

14 a Given that $x^3 - x^2 - 17x - 15 = (x + 3)(x^2 + bx + c)$, where b and c are constants, work out the values of b and c.

b Hence, fully factorise $x^3 - x^2 - 17x - 15$.

E **15** Given that $y = \frac{1}{64}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(1 mark)**

b $4y^{-1}$ **(1 mark)**

E/P **16** Show that $\frac{5}{\sqrt{75} - \sqrt{50}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are integers. **(5 marks)**

E **17** Expand and simplify $(\sqrt{11} - 5)(5 - \sqrt{11})$. **(2 marks)**

E **18** Factorise completely $x - 64x^3$. **(3 marks)**

E/P **19** Express 27^{2x+1} in the form 3^y, stating y in terms of x. **(2 marks)**

E/P **20** Solve the equation $8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$

Give your answer in the form $a\sqrt{b}$ where a and b are integers. **(4 marks)**

P **21** A rectangle has a length of $(1 + \sqrt{3})$ cm and area of $\sqrt{12}$ cm^2.
Calculate the width of the rectangle in cm.
Express your answer in the form $a + b\sqrt{3}$, where a and b are integers to be found.

E **22** Show that $\frac{(2 - \sqrt{x})^2}{\sqrt{x}}$ can be written as $4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$. **(2 marks)**

E/P **23** Given that $243\sqrt{3} = 3^a$, find the value of a. **(3 marks)**

E/P **24** Given that $\frac{4x^3 + x^{\frac{5}{2}}}{\sqrt{x}}$ can be written in the form $4x^a + x^b$, write down the value of a and the value of b. **(2 marks)**

Challenge

a Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{24} + \sqrt{25}} = 4$

Summary of key points

1 You can use the laws of indices to simplify powers of the **same base**.

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$

2 Factorising is the opposite of expanding brackets.

3 A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

4 $x^2 - y^2 = (x + y)(x - y)$

5 You can use the laws of indices with any rational power.

- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$

6 You can manipulate surds using these rules:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

7 The rules to rationalise denominators are:

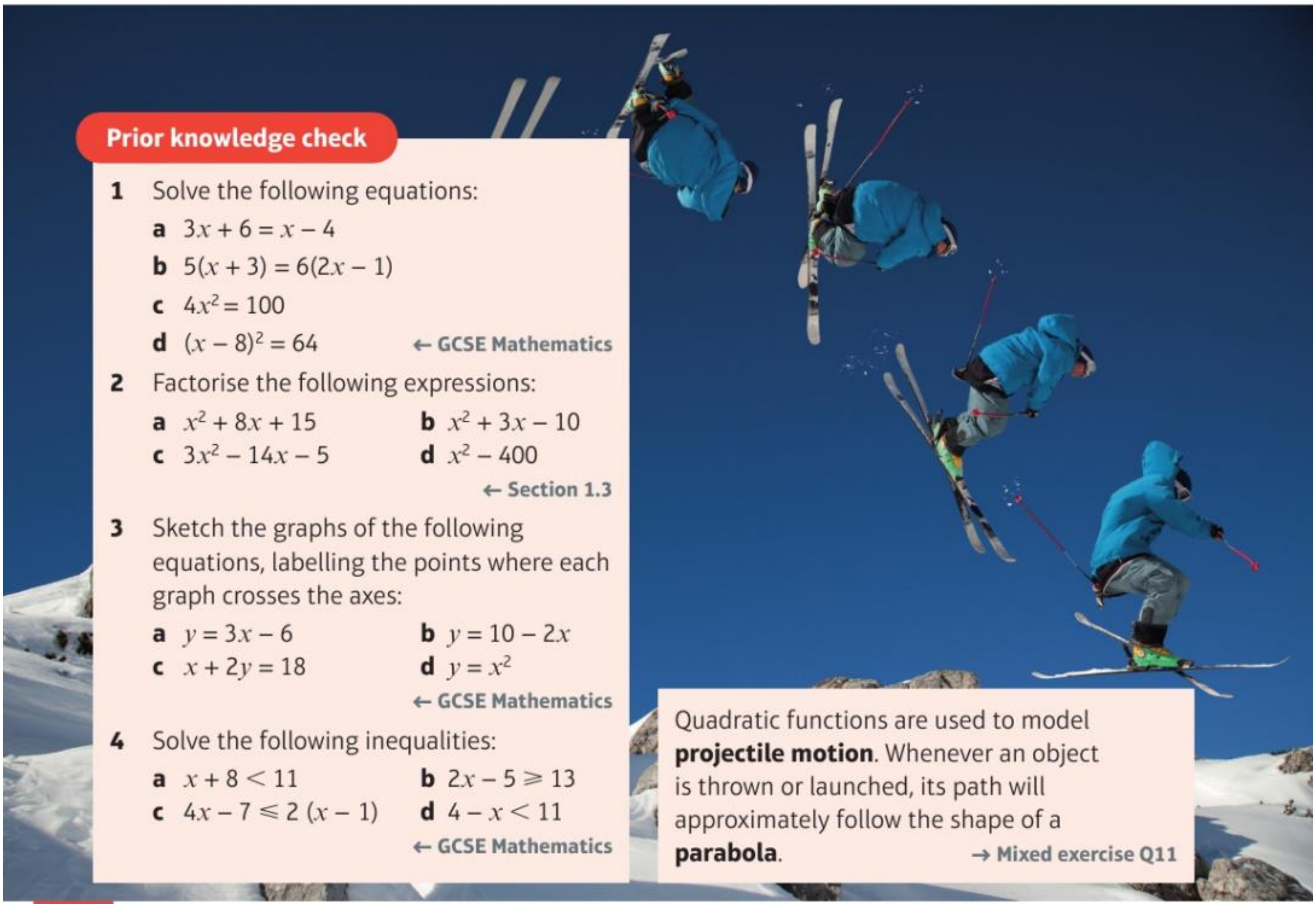
- Fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
- Fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$.
- Fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$.

2 Quadratics

Objectives

After completing this chapter you should be able to:

- Solve quadratic equations using factorisation, the quadratic formula and completing the square → pages 19 – 24
- Read and use f(x) notation when working with functions → pages 25 – 27
- Sketch the graph and find the turning point of a quadratic function → pages 27 – 30
- Find and interpret the discriminant of a quadratic expression → pages 30 – 32
- Use and apply models that involve quadratic functions → pages 32 – 35



Prior knowledge check

1 Solve the following equations:

a $3x + 6 = x - 4$

b $5(x + 3) = 6(2x - 1)$

c $4x^2 = 100$

d $(x - 8)^2 = 64$ ← GCSE Mathematics

2 Factorise the following expressions:

a $x^2 + 8x + 15$ **b** $x^2 + 3x - 10$

c $3x^2 - 14x - 5$ **d** $x^2 - 400$

← Section 1.3

3 Sketch the graphs of the following equations, labelling the points where each graph crosses the axes:

a $y = 3x - 6$ **b** $y = 10 - 2x$

c $x + 2y = 18$ **d** $y = x^2$

← GCSE Mathematics

4 Solve the following inequalities:

a $x + 8 < 11$ **b** $2x - 5 \geqslant 13$

c $4x - 7 \leqslant 2(x - 1)$ **d** $4 - x < 11$

← GCSE Mathematics

Quadratic functions are used to model **projectile motion**. Whenever an object is thrown or launched, its path will approximately follow the shape of a **parabola**. → Mixed exercise Q11

2.1 Solving quadratic equations

A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where a, b and c are real constants, and $a \neq 0$. Quadratic equations can have one, two, or no real solutions.

- **To solve a quadratic equation by factorising:**
  - **Write the equation in the form $ax^2 + bx + c = 0$**
  - **Factorise the left-hand side**
  - **Set each factor equal to zero and solve to find the value(s) of x**

Notation The solutions to an equation are sometimes called the **roots** of the equation.

Example 1

Solve the following equations:

a $x^2 - 2x - 15 = 0$ **b** $x^2 = 9x$

c $6x^2 + 13x - 5 = 0$ **d** $x^2 - 5x + 18 = 2 + 3x$

a $x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$

Factorise the quadratic. ← Section 1.3

Then either $x + 3 = 0 \Rightarrow x = -3$

If the product of the factors is zero, one of the factors must be zero.

Notation The symbol $\Rightarrow$ means 'implies that'. This statement says 'If $x + 3 = 0$, then $x = -3$'.

or $x - 5 = 0 \Rightarrow x = 5$

So $x = -3$ and $x = 5$ are the two solutions of the equation.

A quadratic equation with two distinct factors has two distinct solutions.

Watch out The signs of the solutions are **opposite** to the signs of the constant terms in each factor.

b $x^2 = 9x$

$x^2 - 9x = 0$

Be careful not to divide both sides by x, since x may have the value 0. Instead, rearrange into the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$

Factorise.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

The solutions are $x = 0$ and $x = 9$.

c $6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$

Factorise.

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ and $x = -\frac{5}{2}$

Solutions to quadratic equations do not have to be integers.

The quadratic equation $(px + q)(rx + s) = 0$ will have solutions $x = -\frac{q}{p}$ and $x = -\frac{s}{r}$.

d $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$

Rearrange into the form $ax^2 + bx + c = 0$.

$(x - 4)(x - 4) = 0$

Factorise.

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Notation When a quadratic equation has exactly one root it is called a **repeated root**. You can also say that the equation has two equal roots.

In some cases it may be more straightforward to solve a quadratic equation without factorising.

Example 2

Solve the following equations

a $(2x-3)^2 = 25$ **b** $(x-3)^2 = 7$

Notation The symbol $\pm$ lets you write two statements in one line of working. You say 'plus or minus'.

a $(2x-3)^2 = 25$

$2x - 3 = \pm 5$

$2x = 3 \pm 5$

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ and $x = -1$

Take the square root of both sides. Remember $5^2 = (-5)^2 = 25$.

Add 3 to both sides.

b $(x-3)^2 = 7$

$x - 3 = \pm\sqrt{7}$

$x = 3 \pm \sqrt{7}$

The solutions are $x = 3 + \sqrt{7}$ and $x = 3 - \sqrt{7}$

Take square roots of both sides.

You can leave your answer in surd form.

Exercise 2A

1 Solve the following equations using factorisation:

a $x^2 + 3x + 2 = 0$ **b** $x^2 + 5x + 4 = 0$ **c** $x^2 + 7x + 10 = 0$ **d** $x^2 - x - 6 = 0$

e $x^2 - 8x + 15 = 0$ **f** $x^2 - 9x + 20 = 0$ **g** $x^2 - 5x - 6 = 0$ **h** $x^2 - 4x - 12 = 0$

2 Solve the following equations using factorisation:

a $x^2 = 4x$ **b** $x^2 = 25x$ **c** $3x^2 = 6x$ **d** $5x^2 = 30x$

e $2x^2 + 7x + 3 = 0$ **f** $6x^2 - 7x - 3 = 0$ **g** $6x^2 - 5x - 6 = 0$ **h** $4x^2 - 16x + 15 = 0$

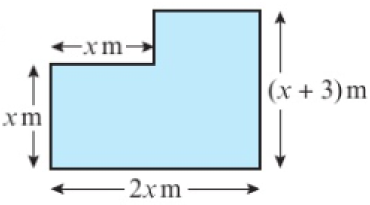
3 Solve the following equations:

a $3x^2 + 5x = 2$ **b** $(2x-3)^2 = 9$ **c** $(x-7)^2 = 36$ **d** $2x^2 = 8$ **e** $3x^2 = 5$

f $(x-3)^2 = 13$ **g** $(3x-1)^2 = 11$ **h** $5x^2 - 10x^2 = -7 + x + x^2$

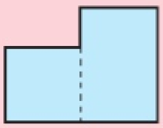
i $6x^2 - 7 = 11x$ **j** $4x^2 + 17x = 6x - 2x^2$

P **4** This shape has an area of $44\,\text{m}^2$. Find the value of x.



Problem-solving

Divide the shape into two sections:



P **5** Solve the equation $5x + 3 = \sqrt{3x + 7}$.

Some equations cannot be easily factorised. You can also solve quadratic equations using the **quadratic formula**.

- **The solutions of the equation $ax^2 + bx + c = 0$ are given by the formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Watch out You need to rearrange the equation into the form $ax^2 + bx + c = 0$ before reading off the coefficients.

Example 3

Solve $3x^2 - 7x - 1 = 0$ by using the formula.

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-1)}}{2 \times 3}$$

$a = 3$, $b = -7$ and $c = -1$.

Put brackets around any negative values.

$$x = \frac{7 \pm \sqrt{49 + 12}}{6}$$

$-4 \times 3 \times (-1) = +12$

$$x = \frac{7 \pm \sqrt{61}}{6}$$

Then $x = \frac{7 + \sqrt{61}}{6}$ or $x = \frac{7 - \sqrt{61}}{6}$

Or $x = 2.47$ (3 s.f.) or $x = -0.135$ (3 s.f.)

Exercise 2B

1 Solve the following equations using the quadratic formula.
Give your answers exactly, leaving them in surd form where necessary.

a $x^2 + 3x + 1 = 0$ **b** $x^2 - 3x - 2 = 0$ **c** $x^2 + 6x + 6 = 0$ **d** $x^2 - 5x - 2 = 0$
e $3x^2 + 10x - 2 = 0$ **f** $4x^2 - 4x - 1 = 0$ **g** $4x^2 - 7x = 2$ **h** $11x^2 + 2x - 7 = 0$

2 Solve the following equations using the quadratic formula.
Give your answers to three significant figures.

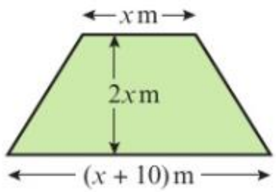
a $x^2 + 4x + 2 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $x^2 + 11x - 9 = 0$ **d** $x^2 - 7x - 17 = 0$
e $5x^2 + 9x - 1 = 0$ **f** $2x^2 - 3x - 18 = 0$ **g** $3x^2 + 8 = 16x$ **h** $2x^2 + 11x = 5x^2 - 18$

3 For each of the equations below, choose a suitable method and find all of the solutions.
Where necessary, give your answers to three significant figures.

a $x^2 + 8x + 12 = 0$ **b** $x^2 + 9x - 11 = 0$
c $x^2 - 9x - 1 = 0$ **d** $2x^2 + 5x + 2 = 0$
e $(2x + 8)^2 = 100$ **f** $6x^2 + 6 = 12x$
g $2x^2 - 11 = 7x$ **h** $x = \sqrt{8x - 15}$

Hint You can use any method you are confident with to solve these equations.

Ⓟ **4** This trapezium has an area of $50\,\text{m}^2$.
Show that the height of the trapezium is equal to $5(\sqrt{5} - 1)\,\text{m}$.



Problem-solving

Height must be positive. You will have to discard the negative solution of your quadratic equation.

Hint Write the equation in the form $ax^2 + bx + c = 0$ before using the quadratic formula or factorising.

Challenge

Given that x is positive, solve the equation
$\frac{1}{x} + \frac{1}{x+2} = \frac{28}{195}$

3 Equations and inequalities

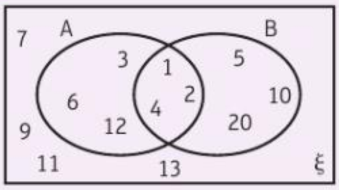
Objectives

After completing this chapter you should be able to:

- Solve linear simultaneous equations using elimination or substitution → pages 39 – 40
- Solve simultaneous equations: one linear and one quadratic → pages 41 – 42
- Interpret algebraic solutions of equations graphically → pages 42 – 45
- Solve linear inequalities → pages 46 – 48
- Solve quadratic inequalities → pages 48 – 51
- Interpret inequalities graphically → pages 51 – 53
- Represent linear and quadratic inequalities graphically → pages 53 – 55

Prior knowledge check

1 A = {factors of 12}
B = {factors of 20}
Write down the numbers in each of these sets:



a $A \cap B$ **b** $(A \cup B)'$

← GCSE Mathematics

2 Simplify these expressions.

a $\sqrt{75}$ **b** $\dfrac{2\sqrt{45} + 3\sqrt{32}}{6}$

← Section 1.5

3.1 Linear simultaneous equations

Linear simultaneous equations in two unknowns have **one set of values** that will make a pair of equations true at the same time.

The solution to this pair of simultaneous equations is $x = 5$, $y = 2$

$x + 3y = 11$ (1) — $5 + 3(2) = 5 + 6 = 11$ ✓

$4x - 5y = 10$ (2) — $4(5) - 5(2) = 20 - 10 = 10$ ✓

- **Linear simultaneous equations can be solved using elimination or substitution.**

Example 1

Solve the simultaneous equations:

a $2x + 3y = 8$
$3x - y = 23$

b $4x - 5y = 4$
$6x + 2y = 25$

a $2x + 3y = 8$ (1)

$3x - y = 23$ (2)

First look for a way to eliminate x or y.

Multiply equation (2) by 3 to get $3y$ in each equation.

$9x - 3y = 69$ (3)

Number this new equation (3).

$11x = 77$

$x = 7$

Then add equations (1) and (3), since the $3y$ terms have different signs and y will be eliminated.

$14 + 3y = 8$

Substitute $x = 7$ into equation (1) to find y.

$3y = 8 - 14$

$y = -2$

The solution is $x = 7$, $y = -2$.

Remember to check your solution by substituting into equation (2). $3(7) - (-2) = 21 + 2 = 23$ ✓

Note that you could also multiply equation (1) by 3 and equation (2) by 2 to get $6x$ in both equations. You could then subtract to eliminate x.

b $4x - 5y = 4$ (1)

$6x + 2y = 25$ (2)

Multiply equation (1) by 3 and multiply equation (2) by 2 to get $12x$ in each equation.

$12x - 15y = 12$ (3)

$12x + 4y = 50$ (4)

$-19y = -38$

Subtract, since the $12x$ terms have the same sign (both positive).

$y = 2$

$4x - 10 = 4$

Substitute $y = 2$ into equation (1) to find x.

$4x = 14$

$x = 3\frac{1}{2}$

The solution is $x = 3\frac{1}{2}$, $y = 2$.

Example 2

Solve the simultaneous equations:

$2x - y = 1$
$4x + 2y = -30$

$2x - y = 1 \quad (1)$
$4x + 2y = -30 \quad (2)$

$y = 2x - 1$

Rearrange an equation, in this case equation (1), to get either $x = \dots$ or $y = \dots$ (here $y = \dots$).

$4x + 2(2x - 1) = -30$

Substitute this into the other equation (here into equation (2) in place of y).

$4x + 4x - 2 = -30$
$8x = -28$
$x = -3\frac{1}{2}$

Solve for x.

$y = 2(-3\frac{1}{2}) - 1 = -8$

Substitute $x = -3\frac{1}{2}$ into equation (1) to find the value of y.

The solution is $x = -3\frac{1}{2}$, $y = -8$

Remember to check your solution in equation (2).
$4(-3.5) + 2(-8) = -14 - 16 = -30$ ✓

Exercise 3A

1 Solve these simultaneous equations by elimination:

a $2x - y = 6$
$4x + 3y = 22$

b $7x + 3y = 16$
$2x + 9y = 29$

c $5x + 2y = 6$
$3x - 10y = 26$

d $2x - y = 12$
$6x + 2y = 21$

e $3x - 2y = -6$
$6x + 3y = 2$

f $3x + 8y = 33$
$6x = 3 + 5y$

2 Solve these simultaneous equations by substitution:

a $x + 3y = 11$
$4x - 7y = 6$

b $4x - 3y = 40$
$2x + y = 5$

c $3x - y = 7$
$10x + 3y = -2$

d $2y = 2x - 3$
$3y = x - 1$

3 Solve these simultaneous equations:

a $3x - 2y + 5 = 0$
$5(x + y) = 6(x + 1)$

b $\dfrac{x - 2y}{3} = 4$
$2x + 3y + 4 = 0$

c $3y = 5(x - 2)$
$3(x - 1) + y + 4 = 0$

Hint First rearrange both equations into the same form e.g. $ax + by = c$.

E/P **4** $3x + ky = 8$
$x - 2ky = 5$
are simultaneous equations where k is a constant.

a Show that $x = 3$. **(3 marks)**

b Given that $y = \frac{1}{2}$ determine the value of k. **(1 mark)**

Problem-solving k is a constant, so it has the same value in both equations.

E/P **5** $2x - py = 5$
$4x + 5y + q = 0$
are simultaneous equations where p and q are constants.
The solution to this pair of simultaneous equations is $x = q$, $y = -1$.
Find the value of p and the value of q. **(5 marks)**

3.2 Quadratic simultaneous equations

You need to be able to solve simultaneous equations where one equation is linear and one is quadratic.

To solve simultaneous equations involving one linear equation and one quadratic equation, you need to use a substitution method from the linear equation into the quadratic equation.

- **Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.**

The solutions to this pair of simultaneous equations are $x = 4$, $y = -3$ and $x = 5.5$, $y = -1.5$.

$x - y = 7$ (1)

$4 - (-3) = 7$ ✓ and $5.5 - (-1.5) = 7$ ✓

$y^2 + xy + 2x = 5$ (2)

$(-3)^2 + (4)(-3) + 2(4) = 9 - 12 + 8 = 5$ ✓ and

$(-1.5)^2 + (5.5)(-1.5) + 2(5.5) = 2.25 - 8.25 + 11 = 5$ ✓

Example 3

Solve the simultaneous equations:

$x + 2y = 3$

$x^2 + 3xy = 10$

$x + 2y = 3$ (1)

$x^2 + 3xy = 10$ (2)

The quadratic equation can contain terms involving y^2 and xy.

$x = 3 - 2y$

Rearrange linear equation (1) to get $x = \dots$ or $y = \dots$ (here $x = \dots$).

$(3 - 2y)^2 + 3y(3 - 2y) = 10$

Substitute this into quadratic equation (2) (here in place of x).

$(3 - 2y)^2$ means $(3 - 2y)(3 - 2y)$ ← Section 1.2

$9 - 12y + 4y^2 + 9y - 6y^2 = 10$

$-2y^2 - 3y - 1 = 0$

$2y^2 + 3y + 1 = 0$

$(2y + 1)(y + 1) = 0$

Solve for y using factorisation.

$y = -\frac{1}{2}$ or $y = -1$

So $x = 4$ or $x = 5$

Find the corresponding x-values by substituting the y-values into linear equation (1), $x = 3 - 2y$.

Solutions are $x = 4$, $y = -\frac{1}{2}$ and $x = 5$, $y = -1$.

There are two solution pairs for x and y.

Exercise 3B

1 Solve the simultaneous equations:

a $x + y = 11$, $xy = 30$

b $2x + y = 1$, $x^2 + y^2 = 1$

c $y = 3x$, $2y^2 - xy = 15$

d $3a + b = 8$, $3a^2 + b^2 = 28$

e $2u + v = 7$, $uv = 6$

f $3x + 2y = 7$, $x^2 + y = 8$

2 Solve the simultaneous equations:

a $2x + 2y = 7$, $x^2 - 4y^2 = 8$

b $x + y = 9$, $x^2 - 3xy + 2y^2 = 0$

c $5y - 4x = 1$, $x^2 - y^2 + 5x = 41$

3 Solve the simultaneous equations, giving your answers in their simplest surd form:

a $x - y = 6$
$xy = 4$

b $2x + 3y = 13$
$x^2 + y^2 = 78$

Watch out Use brackets when you are substituting an expression into an equation.

E/P 4 Solve the simultaneous equations:

$x + y = 3$
$x^2 - 3y = 1$ **(6 marks)**

E/P 5 **a** By eliminating y from the equations

$y = 2 - 4x$
$3x^2 + xy + 11 = 0$

show that $x^2 - 2x - 11 = 0$. **(2 marks)**

b Hence, or otherwise, solve the simultaneous equations

$y = 2 - 4x$
$3x^2 + xy + 11 = 0$

giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers. **(5 marks)**

P 6 One pair of solutions for the simultaneous equations

$y = kx - 5$
$4x^2 - xy = 6$

is $(1, p)$ where k and p are constants.

a Find the values of k and p.

b Find the second pair of solutions for the simultaneous equations.

Problem-solving If $(1, p)$ is a solution, then $x = 1$, $y = p$ satisfies both equations.

Challenge

$y - x = k$
$x^2 + y^2 = 4$

Given that the simultaneous equations have exactly one pair of solutions, show that

$k = \pm 2\sqrt{2}$

3.4 Linear inequalities

You can solve linear inequalities using similar methods to those for solving linear equations.

- **The solution of an inequality is the set of all real numbers x that make the inequality true.**

Example 7

Find the set of values of x for which:

a $5x + 9 \geqslant x + 20$ **b** $12 - 3x < 27$

c $3(x - 5) > 5 - 2(x - 8)$

Notation You can write the solution to this inequality using set notation as $\{x : x \geqslant 2.75\}$. This means the set of all values x for which x is greater than or equal to 2.75.

a $5x + 9 \geqslant x + 20$

$4x + 9 \geqslant 20$

$4x \geqslant 11$

$x \geqslant 2.75$

Rearrange to get $x \geqslant \ldots$

b $12 - 3x < 27$

$-3x < 15$

$x > -5$

Subtract 12 from both sides.

Divide both sides by -3. (You therefore need to turn round the inequality sign.)

In set notation $\{x : x > -5\}$.

c $3(x - 5) > 5 - 2(x - 8)$

$3x - 15 > 5 - 2x + 16$

$5x > 5 + 16 + 15$

$5x > 36$

$x > 7.2$

Multiply out (note: $-2 \times -8 = +16$).

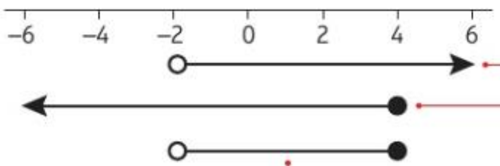
Rearrange to get $x > \ldots$

In set notation $\{x : x > 7.2\}$.

You may sometimes need to find the set of values for which **two** inequalities are true together. Number lines can be useful to find your solution.

Notation In set notation
$x > -2$ **and** $x \leqslant 4$ is written $\{x : -2 < x \leqslant 4\}$
or alternatively $\{x : x > -2\} \cap \{x : x \leqslant 4\}$
$x \leqslant -1$ **or** $x > 3$ is written $\{x : x \leqslant -1\} \cup \{x : x > 3\}$

For example, in the number line below the solution set is $x > -2$ **and** $x \leqslant 4$.

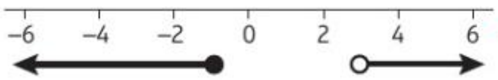


○ is used for < and > and means the end value is not included.

● is used for $\leqslant$ and $\geqslant$ and means the end value is included.

These are the only real values that satisfy both equalities simultaneously so the solution is $-2 < x \leqslant 4$.

Here the solution sets are $x \leqslant -1$ **or** $x > 3$.



Here there is no overlap and the two inequalities have to be written separately as $x \leqslant -1$ or $x > 3$.

Example 8

Find the set of values of x for which:

a $3x - 5 < x + 8$ and $5x > x - 8$

b $x - 5 > 1 - x$ or $15 - 3x > 5 + 2x$.

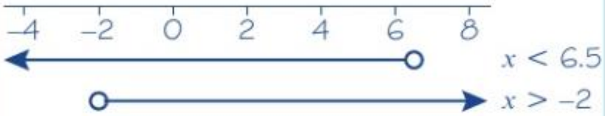
c $4x + 7 > 3$ and $17 < 11 + 2x$.

a $3x - 5 < x + 8$ $\qquad$ $5x > x - 8$

$2x - 5 < 8$ $\qquad$ $4x > -8$

$2x < 13$ $\qquad$ $x > -2$

$x < 6.5$



So the required set of values is $-2 < x < 6.5$.

Draw a number line to illustrate the two inequalities.

The two sets of values overlap (intersect) where $-2 < x < 6.5$.

Notice here how this is written when x lies between two values.

In set notation this can be written as $\{x : -2 < x < 6.5\}$.

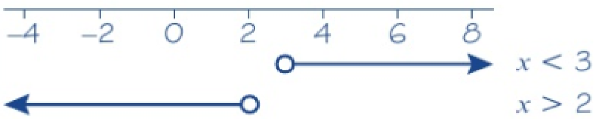
b $x - 5 > 1 - x$ $\qquad$ $15 - 3x > 5 + 2x$

$2x - 5 > 1$ $\qquad$ $10 - 3x > 2x$

$2x > 6$ $\qquad$ $10 > 5x$

$x > 3$ $\qquad$ $2 > x$

$x < 2$



The solution is $x > 3$ or $x < 2$.

Draw a number line. Note that there is no overlap between the two sets of values.

In set notation this can be written as $\{x : x < 2\} \cup \{x : x > 3\}$.

Exercise 3D

1 Find the set of values of x for which:

a $2x - 3 < 5$

b $5x + 4 \geqslant 39$

c $6x - 3 > 2x + 7$

d $5x + 6 \leqslant -12 - x$

e $15 - x > 4$

f $21 - 2x > 8 + 3x$

g $1 + x < 25 + 3x$

h $7x - 7 < 7 - 7x$

i $5 - 0.5x \geqslant 1$

j $5x + 4 > 12 - 2x$

2 Find the set of values of x for which:

a $2(x - 3) \geqslant 0$ **b** $8(1 - x) > x - 1$ **c** $3(x + 7) \leqslant 8 - x$

d $2(x - 3) - (x + 12) < 0$ **e** $1 + 11(2 - x) < 10(x - 4)$ **f** $2(x - 5) \geqslant 3(4 - x)$

g $12x - 3(x - 3) < 45$ **h** $x - 2(5 + 2x) < 11$ **i** $x(x - 4) \geqslant x^2 + 2$

j $x(5 - x) \geqslant 3 + x - x^2$ **k** $3x + 2x(x - 3) \leqslant 2(5 + x^2)$ **l** $x(2x - 5) \leqslant \dfrac{4x(x + 3)}{2} - 9$

3 Use set notation to describe the set of values of x for which:

a $3(x - 2) > x - 4$ and $4x + 12 > 2x + 17$

b $2x - 5 < x - 1$ and $7(x + 1) > 23 - x$

c $2x - 3 > 2$ and $3(x + 2) < 12 + x$

d $15 - x < 2(11 - x)$ and $5(3x - 1) > 12x + 19$

e $3x + 8 \leqslant 20$ and $2(3x - 7) \geqslant x + 6$

f $5x + 3 < 9$ or $5(2x + 1) > 27$

g $4(3x + 7) \leqslant 20$ or $2(3x - 5) \geqslant \dfrac{7 - 6x}{2}$

Challenge

$A = \{x : 3x + 5 > 2\}$ $B = \left\{x : \dfrac{x}{2} + 1 \leqslant 3\right\}$ $C = \{x : 11 < 2x - 1\}$

Given that $A \cap (B \cup C) = \{x : p < x \leqslant q\} \cup \{x : x > r\}$, find the values of p, q and r.

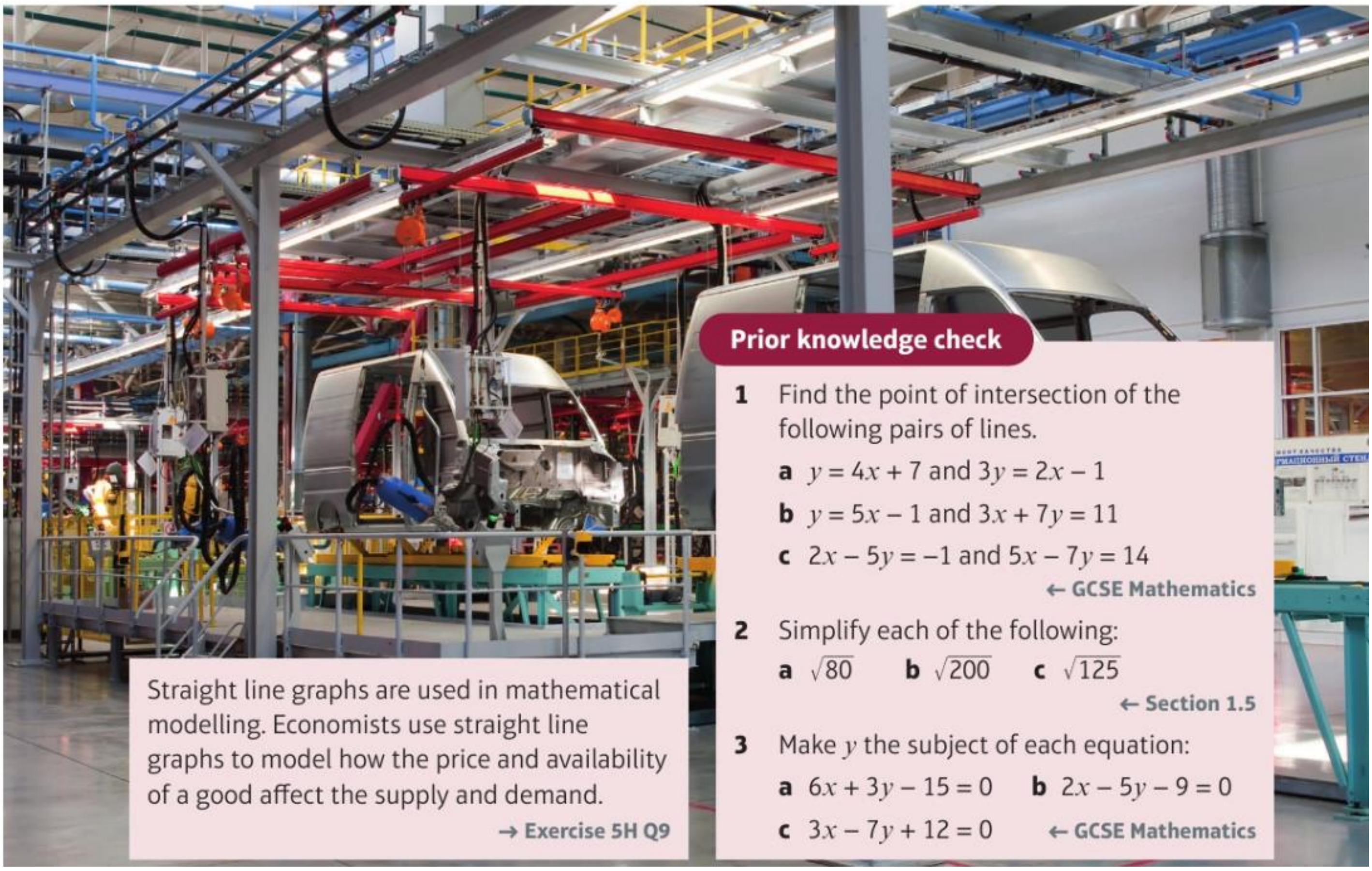
Straight line graphs

5

Objectives

After completing this unit you should be able to:

- Calculate the gradient of a line joining a pair of points → pages 90 – 91
- Understand the link between the equation of a line, and its gradient and intercept → pages 91 – 93
- Find the equation of a line given (i) the gradient and one point on the line or (ii) two points on the line → pages 93 – 95
- Find the point of intersection for a pair of straight lines → pages 95 – 96
- Know and use the rules for parallel and perpendicular gradients → pages 97 – 100
- Solve length and area problems on coordinate grids → pages 100 – 103
- Use straight line graphs to construct mathematical models → pages 103 – 108



Straight line graphs are used in mathematical modelling. Economists use straight line graphs to model how the price and availability of a good affect the supply and demand.
→ Exercise 5H Q9

Prior knowledge check

1 Find the point of intersection of the following pairs of lines.

a $y = 4x + 7$ and $3y = 2x - 1$

b $y = 5x - 1$ and $3x + 7y = 11$

c $2x - 5y = -1$ and $5x - 7y = 14$

← GCSE Mathematics

2 Simplify each of the following:

a $\sqrt{80}$ **b** $\sqrt{200}$ **c** $\sqrt{125}$

← Section 1.5

3 Make y the subject of each equation:

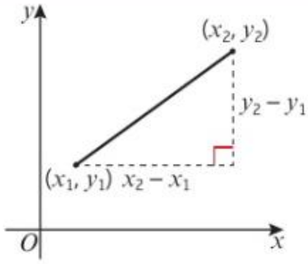
a $6x + 3y - 15 = 0$ **b** $2x - 5y - 9 = 0$

c $3x - 7y + 12 = 0$

← GCSE Mathematics

5.1 $y = mx + c$

You can find the gradient of a straight line joining two points by considering the vertical distance and the horizontal distance between the points.

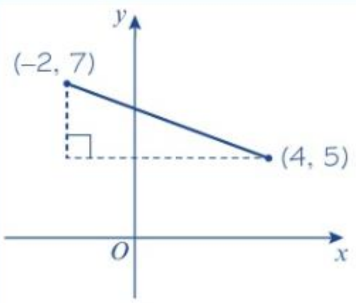


- **The gradient m of a line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) can be calculated using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$**

Online Explore the gradient formula using GeoGebra.

Example 1

Work out the gradient of the line joining (−2, 7) and (4, 5)



$m = \dfrac{5 - 7}{4 - (-2)} = -\dfrac{2}{6} = -\dfrac{1}{3}$

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (-2, 7)$ and $(x_2, y_2) = (4, 5)$

Example 2

The line joining (2, −5) to (4, a) has gradient −1. Work out the value of a.

$\dfrac{a - (-5)}{4 - 2} = -1$

So $\dfrac{a + 5}{2} = -1$

$a + 5 = -2$

$a = -7$

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $m = -1$, $(x_1, y_1) = (2, -5)$ and $(x_2, y_2) = (4, a)$.

Exercise 5A

1 Work out the gradients of the lines joining these pairs of points:

a (4, 2), (6, 3) **b** (−1, 3), (5, 4) **c** (−4, 5), (1, 2)

d (2, −3), (6, 5) **e** (−3, 4), (7, −6) **f** (−12, 3), (−2, 8)

g (−2, −4), (10, 2) **h** $(\frac{1}{2}, 2), (\frac{3}{4}, 4)$ **i** $(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{2}{3})$

j (−2.4, 9.6), (0, 0) **k** (1.3, −2.2), (8.8, −4.7) **l** $(0, 5a), (10a, 0)$

m $(3b, -2b), (7b, 2b)$ **n** $(p, p^2), (q, q^2)$

2 The line joining $(3, -5)$ to $(6, a)$ has a gradient 4. Work out the value of a.

3 The line joining $(5, b)$ to $(8, 3)$ has gradient -3. Work out the value of b.

4 The line joining $(c, 4)$ to $(7, 6)$ has gradient $\frac{3}{4}$. Work out the value of c.

5 The line joining $(-1, 2d)$ to $(1, 4)$ has gradient $-\frac{1}{4}$. Work out the value of d.

6 The line joining $(-3, -2)$ to $(2e, 5)$ has gradient 2. Work out the value of e.

7 The line joining $(7, 2)$ to $(f, 3f)$ has gradient 4. Work out the value of f.

8 The line joining $(3, -4)$ to $(-g, 2g)$ has gradient -3. Work out the value of g.

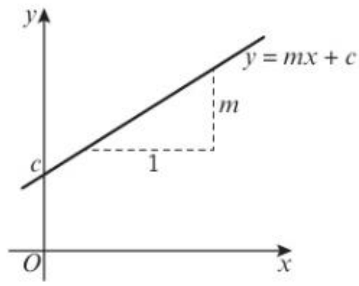
(P) 9 Show that the points $A(2, 3)$, $B(4, 4)$ and $C(10,7)$ can be joined by a straight line.

Problem-solving Find the gradient of the line joining the points A and B and the line joining the points B and C.

(E/P) 10 Show that the points $A(-2a, 5a)$, $B(0, 4a)$ and points $C(6a, a)$ are collinear. **(3 marks)**

Notation Points are collinear if they all lie on the same straight line.

- **The equation of a straight line can be written in the form $y = mx + c$, where m is the gradient and c is the y-intercept.**
- **The equation of a straight line can also be written in the form $ax + by + c = 0$, where a, b and c are integers.**



Example 3

Write down the gradient and y-intercept of these lines:

a $y = -3x + 2$ **b** $4x - 3y + 5 = 0$

a Gradient $= -3$ and y-intercept $= (0, 2)$.

Compare $y = -3x + 2$ with $y = mx + c$. From this, $m = -3$ and $c = 2$.

b $y = \frac{4}{3}x + \frac{5}{3}$

Gradient $= \frac{4}{3}$ and y-intercept $= (0, \frac{5}{3})$.

Rearrange the equation into the form $y = mx + c$. From this $m = \frac{4}{3}$ and $c = \frac{5}{3}$

Watch out Use fractions rather than decimals in coordinate geometry questions.

Example 4

Write these lines in the form $ax + by + c = 0$

a $y = 4x + 3$ **b** $y = -\frac{1}{2}x + 5$

a $4x - y + 3 = 0$

b $\frac{1}{2}x + y - 5 = 0$

$x + 2y - 10 = 0$

Rearrange the equation into the form $ax + by + c = 0$

Collect all the terms on one side of the equation.

Example 5

The line $y = 4x - 8$ meets the x-axis at the point P. Work out the coordinates of P.

$4x - 8 = 0$

$4x = 8$

$x = 2$

So P has coordinates $(2, 0)$

The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 4x - 8$.

Rearrange the equation for x.

Always write down the coordinates of the point.

Exercise 5B

1 Work out the gradients of these lines:

a $y = -2x + 5$ **b** $y = -x + 7$ **c** $y = 4 + 3x$

d $y = \frac{1}{3}x - 2$ **e** $y = -\frac{2}{3}x$ **f** $y = \frac{5}{4}x + \frac{2}{3}$

g $2x - 4y + 5 = 0$ **h** $10x - 5y + 1 = 0$ **i** $-x + 2y - 4 = 0$

j $-3x + 6y + 7 = 0$ **k** $4x + 2y - 9 = 0$ **l** $9x + 6y + 2 = 0$

2 These lines cut the y-axis at $(0, c)$. Work out the value of c in each case.

a $y = -x + 4$ **b** $y = 2x - 5$ **c** $y = \frac{1}{2}x - \frac{2}{3}$

d $y = -3x$ **e** $y = \frac{6}{7}x + \frac{7}{5}$ **f** $y = 2 - 7x$

g $3x - 4y + 8 = 0$ **h** $4x - 5y - 10 = 0$ **i** $-2x + y - 9 = 0$

j $7x + 4y + 12 = 0$ **k** $7x - 2y + 3 = 0$ **l** $-5x + 4y + 2 = 0$

3 Write these lines in the form $ax + by + c = 0$.

a $y = 4x + 3$ **b** $y = 3x - 2$ **c** $y = -6x + 7$

d $y = \frac{4}{5}x - 6$ **e** $y = \frac{5}{3}x + 2$ **f** $y = \frac{7}{3}x$

g $y = 2x - \frac{4}{7}$ **h** $y = -3x + \frac{2}{9}$ **i** $y = -6x - \frac{2}{3}$

j $y = -\frac{1}{3}x + \frac{1}{2}$ **k** $y = \frac{2}{3}x + \frac{5}{6}$ **l** $y = \frac{3}{5}x + \frac{1}{2}$

4 The line $y = 6x - 18$ meets the x-axis at the point P. Work out the coordinates of P.

5 The line $3x + 2y = 0$ meets the x-axis at the point R. Work out the coordinates of R.

6 The line $5x - 4y + 20 = 0$ meets the y-axis at the point A and the x-axis at the point B. Work out the coordinates of A and B.

7 A line l passes through the points with coordinates $(0, 5)$ and $(6, 7)$.

a Find the gradient of the line.

b Find an equation of the line in the form $ax + by + c = 0$.

E 8 A line l cuts the x-axis at $(5, 0)$ and the y-axis at $(0, 2)$.

a Find the gradient of the line. **(1 mark)**

b Find an equation of the line in the form $ax + by + c = 0$. **(2 marks)**

P 9 Show that the line with equation $ax + by + c = 0$ has gradient $-\frac{a}{b}$ and cuts the y-axis at $-\frac{c}{b}$

Problem-solving

Try solving a similar problem with numbers first:

Find the gradient and y-intercept of the straight line with equation $3x + 7y + 2 = 0$.

E/P 10 The line l with gradient 3 and y-intercept $(0, 5)$ has the equation $ax - 2y + c = 0$. Find the values of a and c. **(2 marks)**

E/P 11 The straight line l passes through $(0, 6)$ and has gradient -2. It intersects the line with equation $5x - 8y - 15 = 0$ at point P. Find the coordinates of P. **(4 marks)**

E/P 12 The straight line l_1 with equation $y = 3x - 7$ intersects the straight line l_2 with equation $ax + 4y - 17 = 0$ at the point $P(-3, b)$.

a Find the value of b. **(1 mark)**

b Find the value of a. **(2 marks)**

Challenge

Show that the equation of a straight line through $(0, a)$ and $(b, 0)$ is $ax + by - ab = 0$.

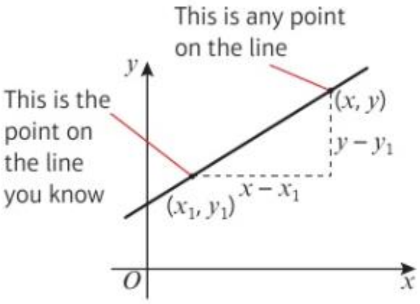
5.2 Equations of straight lines

You can define a straight line by giving:

- one point on the line and the gradient
- two different points on the line

You can find an equation of the line from either of these conditions.

- **The equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) can be written as $y - y_1 = m(x - x_1)$.**



Example 6

Find the equation of the line with gradient 5 that passes through the point (3, 2).

$$y - 2 = 5(x - 3)$$
$$y - 2 = 5x - 15$$
$$y = 5x - 13$$

Online Explore lines of a given gradient passing through a given point using GeoGebra.

This is in the form $y - y_1 = m(x - x_1)$. Here $m = 5$ and $(x_1, y_1) = (3, 2)$.

Example 7

Find the equation of the line that passes through the points (5, 7) and (3, −1).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-1)}{5 - 3} = \frac{8}{2} = 4$$

So $y - y_1 = m(x - x_1)$

$$y + 1 = 4(x - 3)$$
$$y + 1 = 4x - 12$$
$$y = 4x - 13$$

First find the slope of the line.
Here $(x_1, y_1) = (3, -1)$ and $(x_2, y_2) = (5, 7)$.
(x_1, y_1) and (x_2, y_2) have been chosen to make the denominator positive.

You know the gradient and a point on the line, so use $y - y_1 = m(x - x_1)$.
Use $m = 4$, $x_1 = 3$ and $y_1 = -1$.

Exercise 5C

1 Find the equation of the line with gradient m that passes through the point (x_1, y_1) when:

a $m = 2$ and $(x_1, y_1) = (2, 5)$
b $m = 3$ and $(x_1, y_1) = (-2, 1)$
c $m = -1$ and $(x_1, y_1) = (3, -6)$
d $m = -4$ and $(x_1, y_1) = (-2, -3)$
e $m = \frac{1}{2}$ and $(x_1, y_1) = (-4, 10)$
f $m = -\frac{2}{3}$ and $(x_1, y_1) = (-6, -1)$
g $m = 2$ and $(x_1, y_1) = (a, 2a)$
h $m = -\frac{1}{2}$ and $(x_1, y_1) = (-2b, 3b)$

2 Find the equations of the lines that pass through these pairs of points:

a (2, 4) and (3, 8)
b (0, 2) and (3, 5)
c (−2, 0) and (2, 8)
d (5, −3) and (7, 5)
e (3, −1) and (7, 3)
f (−4, −1) and (6, 4)
g (−1, −5) and (−3, 3)
h (−4, −1) and (−3, −9)
i $(\frac{1}{3}, \frac{2}{5})$ and $(\frac{2}{3}, \frac{4}{5})$
j $(-\frac{3}{4}, \frac{1}{7})$ and $(\frac{1}{4}, \frac{3}{7})$

Hint In each case find the gradient m then use $y - y_1 = m(x - x_1)$.

E **3** Find the equation of the line l which passes through the points $A(7, 2)$ and $B(9, -8)$.
Give your answer in the form $ax + by + c = 0$. **(3 marks)**

4 The vertices of the triangle ABC have coordinates $A(3, 5)$, $B(-2, 0)$ and $C(4, -1)$.
Find the equations of the sides of the triangle.

(E/P) **5** The straight line l passes through $(a, 4)$ and $(3a, 3)$. An equation of l is $x + 6y + c = 0$.
Find the value of a and the value of c. **(3 marks)**

Problem-solving

It is often easier to find unknown values in the order they are given in the question. Find the value of a first then find the value of c.

(E/P) **6** The straight line l passes through $(7a, 5)$ and $(3a, 3)$.
An equation of l is $x + by - 12 = 0$.
Find the value of a and the value of b. **(3 marks)**

Challenge

Consider the line passing through points (x_1, y_1) and (x_2, y_2).

a Write down the formula for the gradient, m, of the line.

b Show that the general equation of the line can be written in the form $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

c Use the equation from part **b** to find the equation of the line passing through the points $(-8, 4)$ and $(-1, 7)$.

Example 8

The line $y = 3x - 9$ meets the x-axis at the point A. Find the equation of the line with gradient $\frac{2}{3}$ that passes through point A. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$0 = 3x - 9$ so $x = 3$. A is the point $(3, 0)$.

$$y - 0 = \tfrac{2}{3}(x - 3)$$
$$3y = 2x - 6$$
$$-2x + 3y + 6 = 0$$

Online Plot the solution on a graph using GeoGebra.

The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 3x - 9$.

Use $y - y_1 = m(x - x_1)$. Here $m = \frac{2}{3}$ and $(x_1, y_1) = (3, 0)$.

Rearrange the equation into the form $ax + by + c = 0$.

Example 9

The lines $y = 4x - 7$ and $2x + 3y - 21 = 0$ intersect at the point A. The point B has coordinates $(-2, 8)$. Find the equation of the line that passes through the points A and B. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$$2x + 3(4x - 7) - 21 = 0$$
$$2x + 12x - 21 - 21 = 0$$
$$14x = 42$$
$$x = 3$$

$y = 4(3) - 7 = 5$ so A is the point $(3, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 5}{-2 - 3} = \frac{3}{-5} = -\frac{3}{5}$$
$$y - 5 = -\frac{3}{5}(x - 3)$$
$$5y - 25 = -3x + 9$$
$$3x + 5y - 34 = 0$$

Online Check solutions to simultaneous equations using your calculator.

Solve the equations simultaneously to find point A.
Substitute $y = 4x - 7$ into $2x + 3y - 21 = 0$.

Find the slope of the line connecting A and B.

Use $y - y_1 = m(x - x_1)$ with $m = -\frac{3}{5}$ and $(x_1, y_1) = (3, 5)$.

Exercise 5D

1 The line $y = 4x - 8$ meets the x-axis at the point A. Find the equation of the line with gradient 3 that passes through the point A.

2 The line $y = -2x + 8$ meets the y-axis at the point B. Find the equation of the line with gradient 2 that passes through the point B.

3 The line $y = \frac{1}{2}x + 6$ meets the x-axis at the point C. Find the equation of the line with gradient $\frac{2}{3}$ that passes through the point C. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

P 4 The line $y = \frac{1}{4}x + 2$ meets the y-axis at the point B. The point C has coordinates $(-5, 3)$. Find the gradient of the line joining the points B and C.

P 5 The line that passes through the points $(2, -5)$ and $(-7, 4)$ meets the x-axis at the point P. Work out the coordinates of the point P.

Problem-solving

A sketch can help you check whether your answer looks right.

P 6 The line that passes through the points $(-3, -5)$ and $(4, 9)$ meets the y-axis at the point G. Work out the coordinates of the point G.

P 7 The line that passes through the points $(3, 2\frac{1}{2})$ and $(-1\frac{1}{2}, 4)$ meets the y-axis at the point J. Work out the coordinates of the point J.

P 8 The lines $y = x$ and $y = 2x - 5$ intersect at the point A. Find the equation of the line with gradient $\frac{2}{5}$ that passes through the point A.

P 9 The lines $y = 4x - 10$ and $y = x - 1$ intersect at the point T. Find the equation of the line with gradient $-\frac{2}{3}$ that passes through the point T. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

P 10 The line p has gradient $\frac{2}{3}$ and passes through the point $(6, -12)$. The line q has gradient -1 and passes through the point $(5, 5)$. The line p meets the y-axis at A and the line q meets the x-axis at B. Work out the gradient of the line joining the points A and B.

P 11 The line $y = -2x + 6$ meets the x-axis at the point P. The line $y = \frac{3}{2}x - 4$ meets the y-axis at the point Q. Find the equation of the line joining the points P and Q.

P 12 The line $y = 3x - 5$ meets the x-axis at the point M. The line $y = -\frac{2}{3}x + \frac{2}{3}$ meets the y-axis at the point N. Find the equation of the line joining the points M and N. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

P 13 The line $y = 2x - 10$ meets the x-axis at the point A. The line $y = -2x + 4$ meets the y-axis at the point B. Find the equation of the line joining the points A and B.

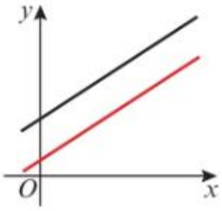
P 14 The line $y = 4x + 5$ meets the y-axis at the point C. The line $y = -3x - 15$ meets the x-axis at the point D. Find the equation of the line joining the points C and D. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

P 15 The lines $y = x - 5$ and $y = 3x - 13$ intersect at the point S. The point T has coordinates $(-4, 2)$. Find the equation of the line that passes through the points S and T.

P 16 The lines $y = -2x + 1$ and $y = x + 7$ intersect at the point L. The point M has coordinates $(-3, 1)$. Find the equation of the line that passes through the points L and M.

5.3 Parallel and perpendicular lines

- **Parallel lines have the same gradient.**



Example 10

A line is parallel to the line $6x + 3y - 2 = 0$ and it passes through the point (0, 3).
Work out the equation of the line.

$6x + 3y - 2 = 0$

$3y - 2 = -6x$

$3y = -6x + 2$

$y = -2x + \frac{2}{3}$

The gradient of this line is −2.

The equation of the line is $y = -2x + 3$.

Rearrange the equation into the form $y = mx + c$ to find m.

Compare $y = -2x + \frac{2}{3}$ with $y = mx + c$, so $m = -2$.
Parallel lines have the same gradient, so the gradient of the required line = −2.

(0, 3) is the intercept on the y-axis, so $c = 3$.

Exercise 5E

1 Work out whether each pair of lines is parallel.

a $y = 5x - 2$
$15x - 3y + 9 = 0$

b $7x + 14y - 1 = 0$
$y = \frac{1}{2}x + 9$

c $4x - 3y - 8 = 0$
$3x - 4y - 8 = 0$

(P) 2 The line r passes through the points (1, 4) and (6, 8) and the line s passes through the points (5, −3) and (20, 9). Show that the lines r and s are parallel.

(P) 3 The coordinates of a quadrilateral $ABCD$ are $A(-6, 2)$, $B(4, 8)$, $C(6, 1)$ and $D(-9, -8)$. Show that the quadrilateral is a trapezium.

Hint A trapezium has exactly one pair of parallel sides.

4 A line is parallel to the line $y = 5x + 8$ and its y-intercept is (0, 3). Write down the equation of the line.

Hint The line will have gradient 5.

5 A line is parallel to the line $y = -\frac{2}{5}x + 1$ and its y-intercept is (0, −4). Work out the equation of the line. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

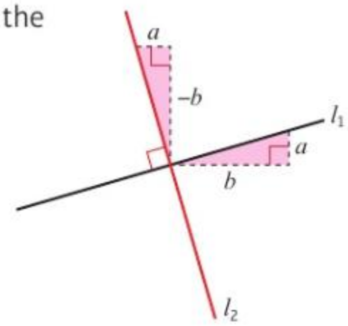
(P) 6 A line is parallel to the line $3x + 6y + 11 = 0$ and its intercept on the y-axis is (0, 7). Write down the equation of the line.

(P) 7 A line is parallel to the line $2x - 3y - 1 = 0$ and it passes through the point (0, 0). Write down the equation of the line.

8 Find an equation of the line that passes through the point (−2, 7) and is parallel to the line $y = 4x + 1$. Write your answer in the form $ax + by + c = 0$.

Perpendicular lines are at right angles to each other. If you know the gradient of one line, you can find the gradient of the other.

- **If a line has a gradient of m, a line perpendicular to it has a gradient of $-\frac{1}{m}$**
- **If two lines are perpendicular, the product of their gradients is -1.**



The shaded triangles are congruent.

Line l_1 has gradient $\frac{a}{b} = m$

Line l_2 has gradient $\frac{-b}{a} = -\frac{1}{m}$

Example 11

Work out whether these pairs of lines are parallel, perpendicular or neither:

a $3x - y - 2 = 0$
$x + 3y - 6 = 0$

b $y = \frac{1}{2}x$
$2x - y + 4 = 0$

a $3x - y - 2 = 0$

$3x - 2 = y$

So $y = 3x - 2$

The gradient of this line is 3.

Rearrange the equations into the form $y = mx + c$.

$x + 3y - 6 = 0$

$3y - 6 = -x$

$3y = -x + 6$

$y = -\frac{1}{3}x + 2$

The gradient of this line is $-\frac{1}{3}$

Compare $y = -\frac{1}{3}x + 2$ with $y = mx + c$, so $m = -\frac{1}{3}$

So the lines are perpendicular as $3 \times (-\frac{1}{3}) = -1$.

b $y = \frac{1}{2}x$

Compare $y = \frac{1}{2}x$ with $y = mx + c$, so $m = \frac{1}{2}$.

The gradient of this line is $\frac{1}{2}$

$2x - y + 4 = 0$

Rearrange the equation into the form $y = mx + c$ to find m.

$2x + 4 = y$

So $y = 2x + 4$

The gradient of this line is 2.

Compare $y = 2x + 4$ with $y = mx + c$, so $m = 2$.

The lines are not parallel as they have different gradients.

The lines are not perpendicular as $\frac{1}{2} \times 2 \neq -1$.

Online Explore this solution using GeoGebra.

Example 12

A line is perpendicular to the line $2y - x - 8 = 0$ and passes through the point $(5, -7)$. Find the equation of the line.

$y = \frac{1}{2}x + 4$

Gradient of $y = \frac{1}{2}x + 4$ is $\frac{1}{2}$

So the gradient of the perpendicular line is -2.

$y - y_1 = m(x - x_1)$

$y + 7 = -2(x - 5)$

$y + 7 = -2x + 10$

$y = -2x + 3$

Problem-solving

You need to fill in the steps of this problem yourself:

- Rearrange the equation into the form $y = mx + c$ to find the gradient.
- Use $-\frac{1}{m}$ to find the gradient of a perpendicular line.
- Use $y - y_1 = m(x - x_1)$ to find the equation of the line.

Exercise 5F

1 Work out whether these pairs of lines are parallel, perpendicular or neither:

a $y = 4x + 2$
$y = -\frac{1}{4}x - 7$

b $y = \frac{2}{3}x - 1$
$y = \frac{2}{3}x - 11$

c $y = \frac{1}{5}x + 9$
$y = 5x + 9$

d $y = -3x + 2$
$y = \frac{1}{3}x - 7$

e $y = \frac{3}{5}x + 4$
$y = -\frac{5}{3}x - 1$

f $y = \frac{5}{7}x$
$y = \frac{5}{7}x - 3$

g $y = 5x - 3$
$5x - y + 4 = 0$

h $5x - y - 1 = 0$
$y = -\frac{1}{5}x$

i $y = -\frac{3}{2}x + 8$
$2x - 3y - 9 = 0$

j $4x - 5y + 1 = 0$
$8x - 10y - 2 = 0$

k $3x + 2y - 12 = 0$
$2x + 3y - 6 = 0$

l $5x - y + 2 = 0$
$2x + 10y - 4 = 0$

2 A line is perpendicular to the line $y = 6x - 9$ and passes through the point $(0, 1)$. Find an equation of the line.

(P) 3 A line is perpendicular to the line $3x + 8y - 11 = 0$ and passes through the point $(0, -8)$. Find an equation of the line.

4 Find an equation of the line that passes through the point $(6, -2)$ and is perpendicular to the line $y = 3x + 5$.

5 Find an equation of the line that passes through the point $(-2, 5)$ and is perpendicular to the line $y = 3x + 6$.

(P) 6 Find an equation of the line that passes through the point $(3, 4)$ and is perpendicular to the line $4x - 6y + 7 = 0$.

7 Find an equation of the line that passes through the point $(5, -5)$ and is perpendicular to the line $y = \frac{2}{3}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

8 Find an equation of the line that passes through the point $(-2, -3)$ and is perpendicular to the line $y = -\frac{4}{7}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) **9** The line l passes through the points $(-3, 0)$ and $(3, -2)$ and the line n passes through the points $(1, 8)$ and $(-1, 2)$. Show that the lines l and n are perpendicular.

Problem-solving
Don't do more work than you need to. You only need to find the gradients of both lines, not their equations.

(P) **10** The vertices of a quadrilateral $ABCD$ have coordinates $A(-1, 5)$, $B(7, 1)$, $C(5, -3)$ and $D(-3, 1)$.
Show that the quadrilateral is a rectangle.

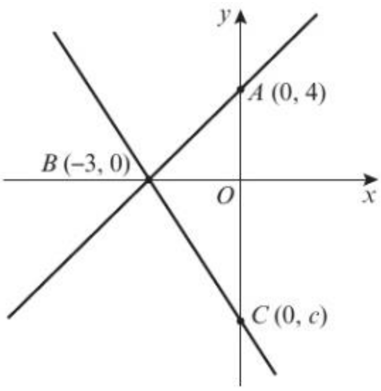
Hint The sides of a rectangle are perpendicular.

(E/P) **11** A line l_1 has equation $5x + 11y - 7 = 0$ and crosses the x-axis at A. The line l_2 is perpendicular to l_1 and passes through A.

a Find the coordinates of the point A. **(1 mark)**

b Find the equation of the line l_2. Write your answer in the form $ax + by + c = 0$. **(3 marks)**

(E/P) **12** The points A and C lie on the y-axis and the point B lies on the x-axis as shown in the diagram.



Problem-solving
Sketch graphs in coordinate geometry problems are not accurate, but you can use the graph to make sure that your answer makes sense. In this question c must be negative.

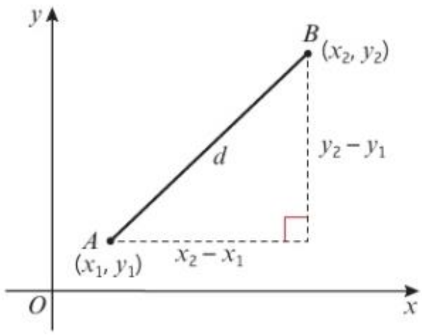
The line through points A and B is perpendicular to the line through points B and C.
Find the value of c. **(6 marks)**

5.4 Length and area

You can find the distance between two points A and B by considering a right-angled triangle with hypotenuse AB.

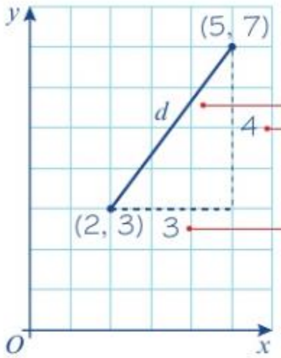
- **You can find the distance d between (x_1, y_1) and (x_2, y_2) by using the formula**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 13

Find the distance between (2, 3) and (5, 7).



Draw a sketch.

Let the distance between the points be d.

The difference in the y-coordinates is $7 - 3 = 4$.

The difference in the x-coordinates is $5 - 2 = 3$.

$d^2 = (5 - 2)^2 + (7 - 3)^2$

$d^2 = 3^2 + 4^2$

$d = \sqrt{(3^2 + 4^2)}$

$= \sqrt{25}$

$= 5$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 7)$.

Example 14

Online Draw both lines and the triangle AOB on a graph using technology.

The straight line l_1 with equation $4x - y = 0$ and the straight line l_2 with equation $2x + 3y - 21 = 0$ intersect at point A.

a Work out the coordinates of A.

b Work out the area of triangle AOB where B is the point where l_2 meets the x-axis.

a Equation of l_1 is $y = 4x$

Rewrite the equation of l_1 in the form $y = mx + c$.

$2x + 3y - 21 = 0$

$2x + 3(4x) - 21 = 0$

Substitute $y = 4x$ into the equation for l_2 to find the point of intersection.

$14x - 21 = 0$

$14x = 21$

$x = \frac{3}{2}$

Solve the equation to find the x-coordinate of point A.

$y = 4 \times \left(\frac{3}{2}\right) = 6$

Substitute to find the y-coordinate of point A.

So point A has coordinates $\left(\frac{3}{2}, 6\right)$.

b The triangle AOB has a height of 6 units.

The height is the y-coordinate of point A.

$2x + 3y - 21 = 0$

$2x + 3(0) - 21 = 0$

B is the point where the line l_2 intersects the x-axis. At B, the y-coordinate is zero.

$2x - 21 = 0$

$x = \frac{21}{2}$

Solve the equation to find the x-coordinate of point B.

The triangle AOB has a base length of $\frac{21}{2}$ units.

Area $= \frac{1}{2} \times 6 \times \frac{21}{2} = \frac{63}{2}$

Area $= \frac{1}{2} \times$ base $\times$ height
You don't need to give units for length and area problems on coordinate grids.

Exercise 5G

1 Find the distance between these pairs of points:

a $(0, 1), (6, 9)$ b $(4, -6), (9, 6)$ c $(3, 1), (-1, 4)$

d $(3, 5), (4, 7)$ e $(0, -4), (5, 5)$ f $(-2, -7), (5, 1)$

2 Consider the points $A(-3, 5)$, $B(-2, -2)$ and $C(3, -7)$. Determine whether the line joining the points A and B is congruent to the line joining the points B and C.

Hint Two line segments are congruent if they are the same length.

3 Consider the points $P(11, -8)$, $Q(4, -3)$ and $R(7, 5)$. Show that the line segment joining the points P and Q is not congruent to the line joining the points Q and R.

(P) 4 The distance between the points $(-1, 13)$ and $(x, 9)$ is $\sqrt{65}$. Find two possible values of x.

Problem-solving Use the distance formula to formulate a quadratic equation in x.

(P) 5 The distance between the points $(2, y)$ and $(5, 7)$ is $3\sqrt{10}$. Find two possible values of y.

(P) 6 a Show that the straight line l_1 with equation $y = 2x + 4$ is parallel to the straight line l_2 with equation $6x - 3y - 9 = 0$.

b Find the equation of the straight line l_3 that is perpendicular to l_1 and passes through the point $(3, 10)$.

c Find the point of intersection of the lines l_2 and l_3.

d Find the shortest distance between lines l_1 and l_2.

Problem-solving The shortest distance between two parallel lines is the perpendicular distance between them.

(E/P) 7 A point P lies on the line with equation $y = 4 - 3x$. The point P is a distance $\sqrt{34}$ from the origin. Find the two possible positions of point P. **(5 marks)**

(P) 8 The vertices of a triangle are $A(2, 7)$, $B(5, -6)$ and $C(8, -6)$.

a Show that the triangle is a scalene triangle.

b Find the area of the triangle ABC.

Notation Scalene triangles have three sides of different lengths.

Problem-solving Draw a sketch and label the points A, B and C. Find the length of the base and the height of the triangle.

9 The straight line l_1 has equation $y = 7x - 3$. The straight line l_2 has equation $4x + 3y - 41 = 0$. The lines intersect at the point A.

a Work out the coordinates of A.

The straight line l_2 crosses the x-axis at the point B.

b Work out the coordinates of B.

c Work out the area of triangle AOB.

10 The straight line l_1 has equation $4x - 5y - 10 = 0$ and intersects the x-axis at point A. The straight line l_2 has equation $4x - 2y + 20 = 0$ and intersects the x-axis at the point B.

a Work out the coordinates of A.

b Work out the coordinates of B.

The straight lines l_1 and l_2 intersect at the point C.

c Work out the coordinates of C.

d Work out the area of triangle ABC.

E **11** The points $R(5, -2)$ and $S(9, 0)$ lie on the straight line l_1 as shown.

a Work out an equation for straight line l_1. **(2 marks)**

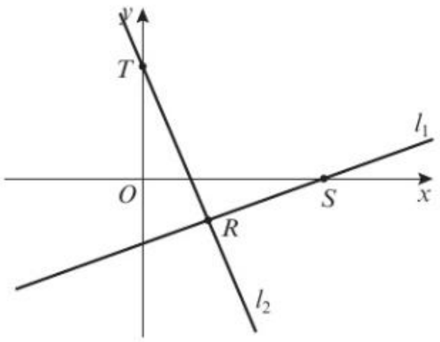
The straight line l_2 is perpendicular to l_1 and passes through the point R.

b Work out an equation for straight line l_2. **(2 marks)**

c Write down the coordinates of T. **(1 mark)**

d Work out the lengths of RS and TR leaving your answer in the form $k\sqrt{5}$. **(2 marks)**

e Work out the area of $\triangle RST$. **(2 marks)**



E/P **12** The straight line l_1 passes through the point $(-4, 14)$ and has gradient $-\frac{1}{4}$

a Find an equation for l_1 in the form $ax + by + c = 0$, where a, b and c are integers. **(3 marks)**

b Write down the coordinates of A, the point where straight line l_1 crosses the y-axis. **(1 mark)**

The straight line l_2 passes through the origin and has gradient 3. The lines l_1 and l_2 intersect at the point B.

c Calculate the coordinates of B. **(2 marks)**

d Calculate the exact area of $\triangle OAB$. **(2 marks)**

Trigonometric ratios

9

Objectives

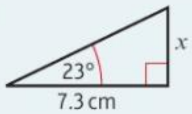
After completing this unit you should be able to:

- Use the cosine rule to find a missing side or angle → **pages 174–179**
- Use the sine rule to find a missing side or angle → **pages 179–185**
- Find the area of a triangle using an appropriate formula → **pages 185–187**
- Solve problems involving triangles → **pages 187–192**
- Sketch the graphs of the sine, cosine and tangent functions → **pages 192–194**
- Sketch simple transformations of these graphs → **pages 194–198**

Prior knowledge check

1 Use trigonometry to find the lengths of the marked sides.

a



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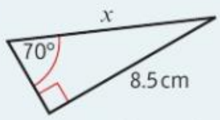
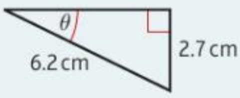


←GCSE Mathematics

2 Find the sizes of the angles marked.

a



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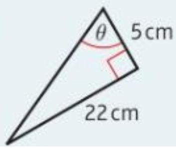
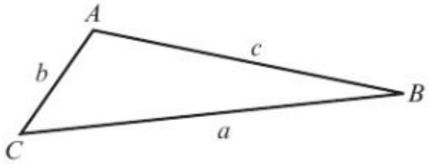


9.1 The cosine rule

The cosine rule can be used to work out missing sides or angles in triangles.

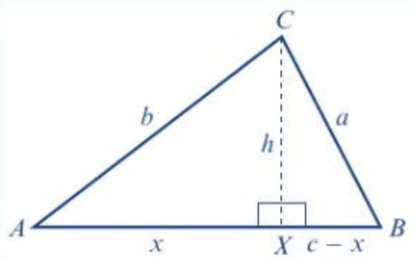
- **This version of the cosine rule is used to find a missing side if you know two sides and the angle between them:**

$$a^2 = b^2 + c^2 - 2bc\cos A$$

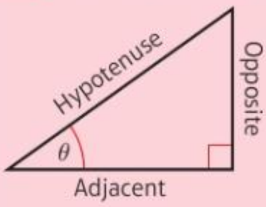


Watch out You can exchange the letters depending on which side you want to find, as long as each side has the same letter as the **opposite** angle.

You can use the standard trigonometric ratios for right-angled triangles to prove the cosine rule:



Hint For a right-angled triangle



$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

$h^2 + x^2 = b^2$ — Use Pythagoras' theorem in $\triangle CAX$.

and $h^2 + (c - x)^2 = a^2$ — Use Pythagoras' theorem in $\triangle CBX$.

So $x^2 - (c - x)^2 = b^2 - a^2$ — Subtract the two equations.

So $2cx - c^2 = b^2 - a^2$ — $(c - x)^2 = c^2 - 2cx + x^2$. So $x^2 - (c - x)^2 = x^2 - c^2 + 2cx - x^2$.

$a^2 = b^2 + c^2 - 2cx$ (1) — Rearrange.

but $x = b\cos A$ (2) — Use the cosine ratio $\cos A = \dfrac{x}{b}$ in $\triangle CAX$.

So $a^2 = b^2 + c^2 - 2bc\cos A$ — Combine (1) and (2). This is the cosine rule.

If you are given all three sides and asked to find an angle, the cosine rule can be rearranged.

$$a^2 + 2bc\cos A = b^2 + c^2$$

$$2bc\cos A = b^2 + c^2 - a^2$$

Hence $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

You can exchange the letters depending on which angle you want to find.

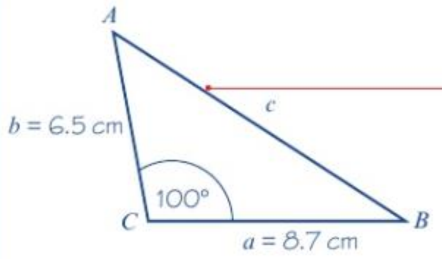
- **This version of the cosine rule is used to find an angle if you know all three sides:**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Online Explore the cosine rule using GeoGebra.

Example 1

Calculate the length of the side AB of the triangle ABC in which $AC = 6.5$ cm, $BC = 8.7$ cm and $\angle ACB = 100°$.



Label the sides of the triangle with small letters a, b and c opposite the angles marked.

$c^2 = a^2 + b^2 - 2ab \cos C$

Write out the formula you are using as the first line of working, then substitute in the values given.

$c^2 = 8.7^2 + 6.5^2 - 2 \times 8.7 \times 6.5 \times \cos 100°$

$= 75.69 + 42.25 - (-19.639...)$

$= 137.57...$

Don't round any values until the end of your working. You can write your final answer to 3 significant figures.

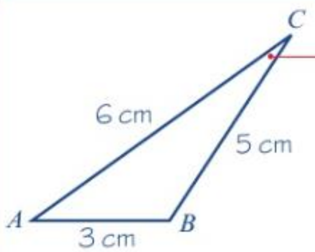
So $c = 11.729...$

Find the square root.

So $AB = 11.7$ cm (3 s.f.)

Example 2

Find the size of the smallest angle in a triangle whose sides have lengths 3 cm, 5 cm and 6 cm.



Label the triangle ABC.
The smallest angle is opposite the smallest side so angle C is required.

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Use the cosine rule $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}$

$= 0.8666...$

Online Use your calculator to work this out efficiently.

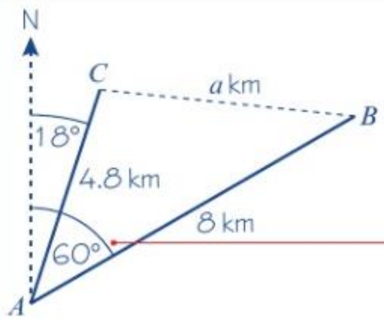
$C = 29.9°$ (3 s.f.)

$C = \cos^{-1}(0.8666...)$

The size of the smallest angle is 29.9°.

Example 3

Coastguard station B is 8 km, on a bearing of 060°, from coastguard station A. A ship C is 4.8 km, on a bearing of 018°, away from A. Calculate how far C is from B.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 4.8^2 + 8^2 - 2 \times 4.8 \times 8 \times \cos 42°$

$= 29.966\ldots$

$a = 5.47$ (3 s.f.)

C is 5.47 km away from coastguard B.

Problem-solving

If no diagram is given with a question you should draw one carefully. Double-check that the information given in the question matches your sketch.

In $\triangle ABC$, $\angle CAB = 60° - 18° = 42°$.

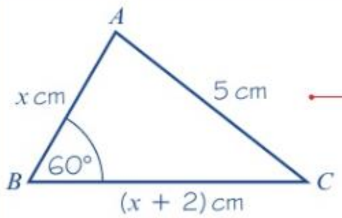
You now have $b = 4.8$ km, $c = 8$ km and $A = 42°$. Use the cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$.

If possible, work this out in one go using your calculator.

Take the square root of 29.966… and round your final answer to 3 significant figures.

Example 4

In $\triangle ABC$, $AB = x$ cm, $BC = (x + 2)$ cm, $AC = 5$ cm and $\angle ABC = 60°$. Find the value of x.



Use the information given in the question to draw a sketch.

$b^2 = a^2 + c^2 - 2ac\cos B$

$5^2 = (x + 2)^2 + x^2 - 2x(x + 2)\cos 60°$

So $25 = 2x^2 + 4x + 4 - x^2 - 2x$

So $x^2 + 2x - 21 = 0$

$x = \dfrac{-2 \pm \sqrt{88}}{2}$

$= 3.69$ (3 s.f.)

Carefully expand and simplify the right-hand side. Note that $\cos 60° = \frac{1}{2}$

Rearrange to the form $ax^2 + bx + c = 0$.

Solve the quadratic equation using the quadratic formula. ← Section 2.1

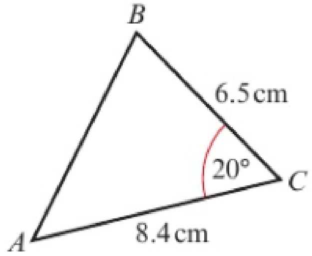
x represents a length so it cannot be negative.

Exercise 9A

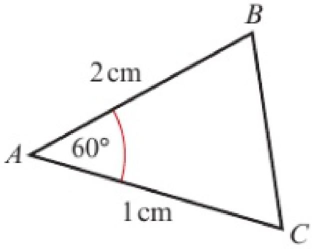
Give answers to 3 significant figures, where appropriate.

1 In each of the following triangles calculate the length of the missing side.

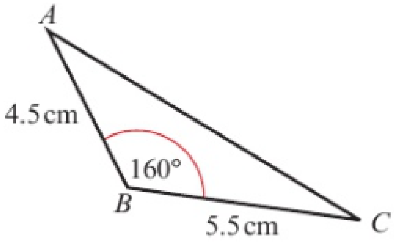
a



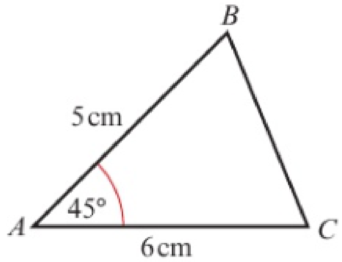
b



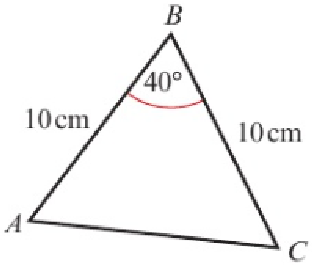
c



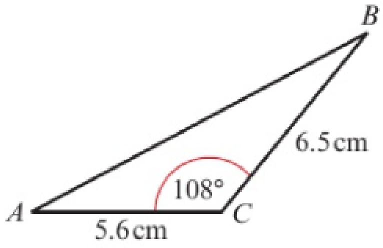
d



e

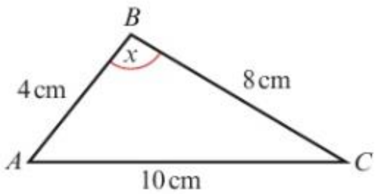


f

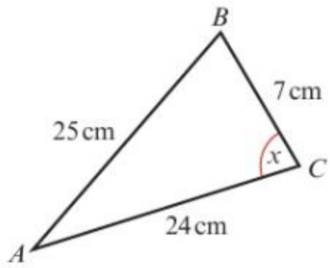


2 In the following triangles calculate the size of the angle marked x:

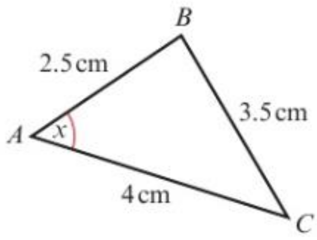
a



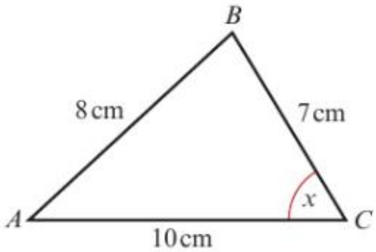
b



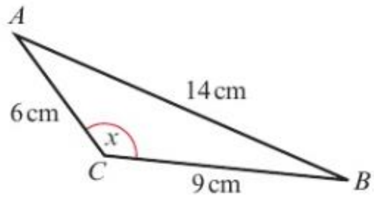
c



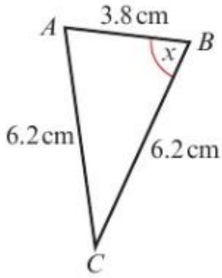
d



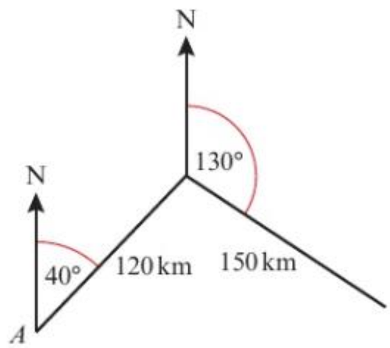
e



f



3 A plane flies from airport A on a bearing of 040° for 120 km and then on a bearing of 130° for 150 km. Calculate the distance of the plane from the airport.

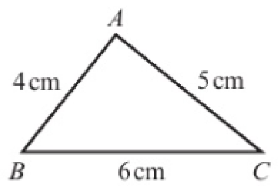


4 From a point A a boat sails due north for 7 km to B. The boat leaves B and moves on a bearing of 100° for 10 km until it reaches C. Calculate the distance of C from A.

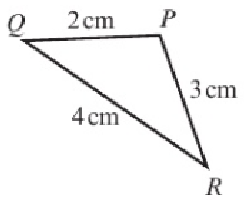
5 A helicopter flies on a bearing of 080° from A to B, where $AB = 50$ km.
It then flies for 60 km to a point C.
Given that C is 80 km from A, calculate the bearing of C from A.

6 The distance from the tee, T, to the flag, F, on a particular hole on a golf course is 494 yards.
A golfer's tee shot travels 220 yards and lands at the point S, where $\angle STF = 22°$.
Calculate how far the ball is from the flag.

(P) **7** Show that $\cos A = \frac{1}{8}$



(P) **8** Show that $\cos P = -\frac{1}{4}$



9 In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $AC = 10$ cm.
Calculate the size of the smallest angle.

10 In $\triangle ABC$, $AB = 9.3$ cm, $BC = 6.2$ cm and $AC = 12.7$ cm.
Calculate the size of the largest angle.

(P) **11** The lengths of the sides of a triangle are in the ratio $2:3:4$.
Calculate the size of the largest angle.

12 In $\triangle ABC$, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $\angle BAC = 60°$.
Use the cosine rule to find the value of x.

(P) **13** In $\triangle ABC$, $AB = x$ cm, $BC = (x - 4)$ cm, $AC = 10$ cm and $\angle BAC = 60°$.
Calculate the value of x.

(P) **14** In $\triangle ABC$, $AB = (5 - x)$ cm, $BC = (4 + x)$ cm, $\angle ABC = 120°$ and $AC = y$ cm.

a Show that $y^2 = x^2 - x + 61$.

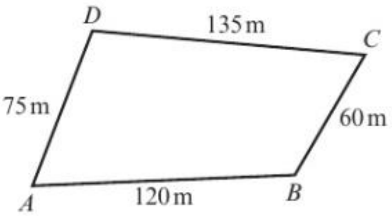
b Use the method of completing the square to find the minimum value of y^2, and give the value of x for which this occurs.

P **15** In $\triangle ABC$, $AB = x$ cm, $BC = 5$ cm, $AC = (10 - x)$ cm.

a Show that $\cos \angle ABC = \dfrac{4x - 15}{2x}$

b Given that $\cos \angle ABC = -\frac{1}{7}$, work out the value of x.

P **16** A farmer has a field in the shape of a quadrilateral as shown.



Problem-solving

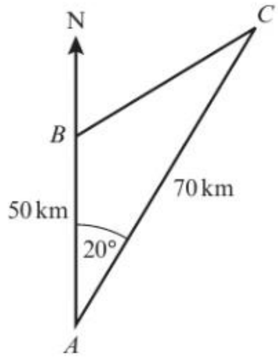
You will have to use the cosine rule twice. Copy the diagram and write any angles or lengths you work out on your copy.

The angle between fences AB and AD is 74°. Find the angle between fences BC and CD.

E/P **17** The diagram shows three cargo ships, A, B and C, which are in the same horizontal plane. Ship B is 50 km due north of ship A and ship C is 70 km from ship A. The bearing of C from A is 020°.

a Calculate the distance between ships B and C, in kilometres to 3 s.f. **(3 marks)**

b Calculate the bearing of ship C from ship B. **(4 marks)**

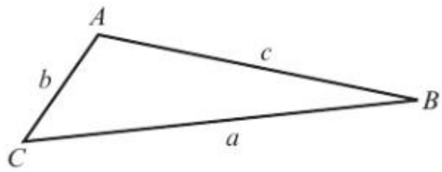


9.2 The sine rule

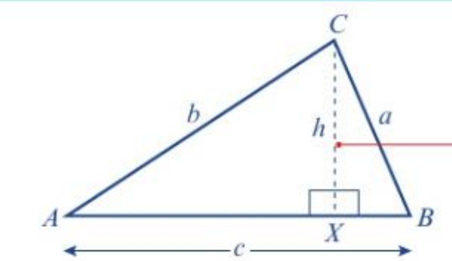
The sine rule can be used to work out missing sides or angles in triangles.

- **This version of the sine rule is used to find the length of a missing side:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



You can use the standard trigonometric ratios for right-angled triangles to prove the sine rule:



$\sin B = \dfrac{h}{a} \Rightarrow h = a \sin B$

In a general triangle ABC, draw the perpendicular from C to AB. It meets AB at X.

The length of CX is h.

Use the sine ratio in triangle CBX.

Online Explore the sine rule using GeoGebra.

and $\sin A = \frac{h}{b} \Rightarrow h = b \sin A$

Use the sine ratio in triangle CAX.

So $a \sin B = b \sin A$

So $\frac{a}{\sin A} = \frac{b}{\sin B}$

Divide throughout by $\sin A \sin B$.

In a similar way, by drawing the perpendicular from B to the side AC, you can show that:

$\frac{a}{\sin A} = \frac{c}{\sin C}$

So $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

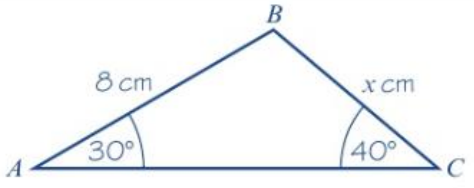
This is the sine rule and is true for all triangles.

- **This version of the sine rule is used to find a missing angle:**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 5

In $\triangle ABC$, $AB = 8$ cm, $\angle BAC = 30°$ and $\angle BCA = 40°$. Find BC.



Always draw a diagram and carefully add the data. Here $c = 8$ (cm), $C = 40°$, $A = 30°$, $a = x$ (cm).

In a triangle, the larger a side is, the larger the opposite angle is. Here, as $C > A$, then $c > a$, so you know that $8 > x$.

$\frac{a}{\sin A} = \frac{c}{\sin C}$

Use this version of the sine rule to find a missing side. Write the formula you are going to use as the first line of working.

$\frac{x}{\sin 30°} = \frac{8}{\sin 40°}$

So $x = \frac{8 \sin 30°}{\sin 40°} = 6.2228...$

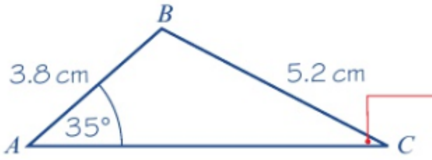
Multiply throughout by $\sin 30°$.

$= 6.22$ cm (3 s.f.)

Give your answer to 3 significant figures.

Example 6

In $\triangle ABC$, $AB = 3.8$ cm, $BC = 5.2$ cm and $\angle BAC = 35°$. Find $\angle ABC$.



Here $a = 5.2$ cm, $c = 3.8$ cm and $A = 35°$. You first need to find angle C.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{3.8} = \frac{\sin 35°}{5.2}$$

So $\sin C = \dfrac{3.8 \sin 35°}{5.2}$

$C = 24.781\ldots$

So $B = 120°$ (3 s.f.)

Use $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

Write the formula you are going to use as the first line of working.

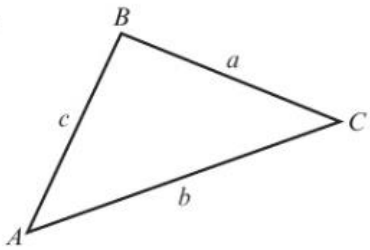
Use your calculator to find the value of C in a single step. Don't round your answer at this point.

$B = 180° - (24.781\ldots° + 35°) = 120.21\ldots$ which rounds to 120° to 3 s.f.

Exercise 9B

Give answers to 3 significant figures, where appropriate.

1 In each of parts **a** to **d**, the given values refer to the general triangle.



a Given that $a = 8$ cm, $A = 30°$, $B = 72°$, find b.

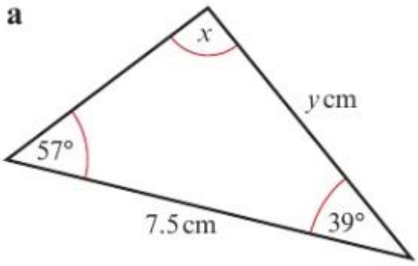
b Given that $a = 24$ cm, $A = 110°$, $C = 22°$, find c.

c Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find a.

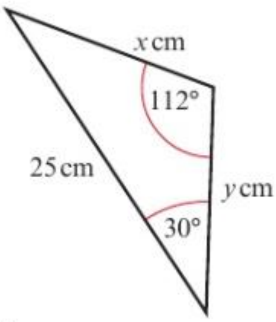
d Given that $c = 9.8$ cm, $B = 68.4°$, $C = 83.7°$, find a.

2 In each of the following triangles calculate the values of x and y.

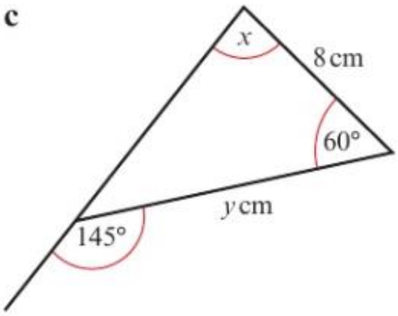
a



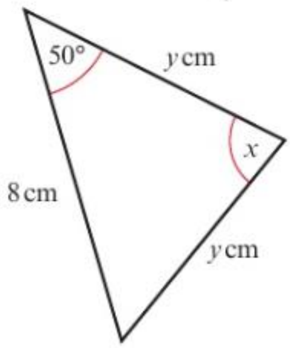
b



c

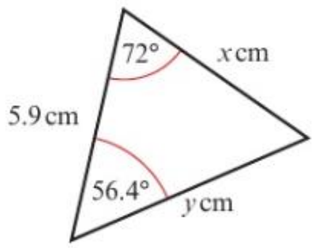


d

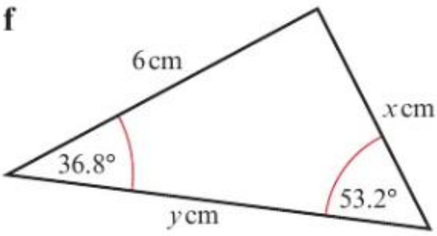


Hint In parts **c** and **d**, start by finding the size of the third angle.

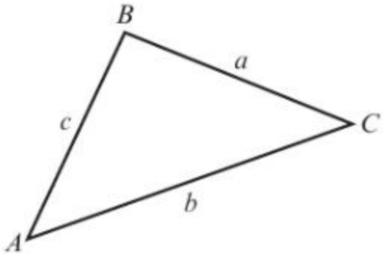
e



f



3 In each of the following sets of data for a triangle ABC, find the value of x.



a $AB = 6\text{ cm}$, $BC = 9\text{ cm}$, $\angle BAC = 117°$, $\angle ACB = x$

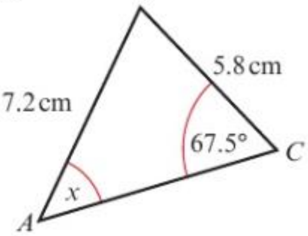
b $AC = 11\text{ cm}$, $BC = 10\text{ cm}$, $\angle ABC = 40°$, $\angle CAB = x$

c $AB = 6\text{ cm}$, $BC = 8\text{ cm}$, $\angle BAC = 60°$, $\angle ACB = x$

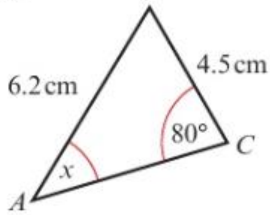
d $AB = 8.7\text{ cm}$, $AC = 10.8\text{ cm}$, $\angle ABC = 28°$, $\angle BAC = x$

4 In each of the diagrams shown below, work out the size of angle x.

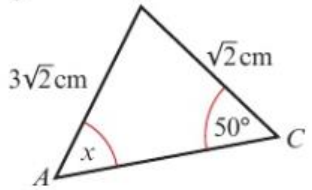
a



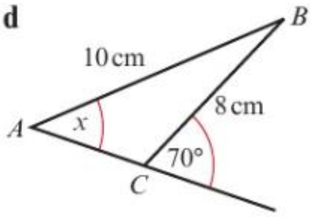
b



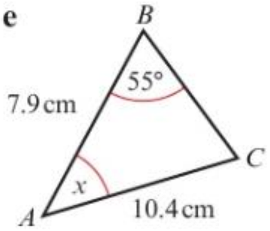
c



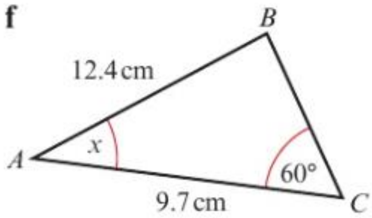
d



e



f

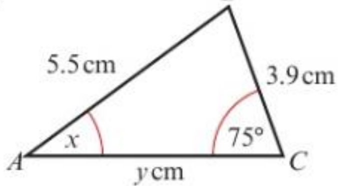


5 In $\triangle PQR$, $QR = \sqrt{3}\text{ cm}$, $\angle PQR = 45°$ and $\angle QPR = 60°$. Find **a** PR and **b** PQ.

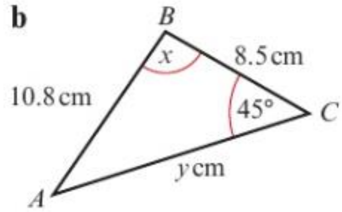
6 In $\triangle PQR$, $PQ = 15\text{ cm}$, $QR = 12\text{ cm}$ and $\angle PRQ = 75°$. Find the two remaining angles.

7 In each of the following diagrams work out the values of x and y.

a



b



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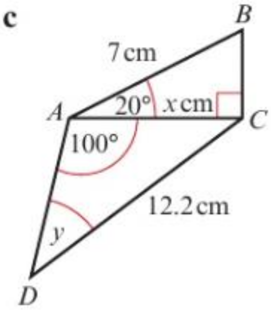

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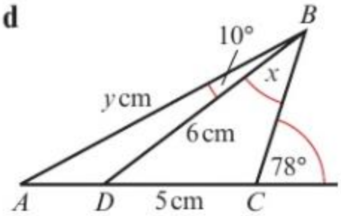
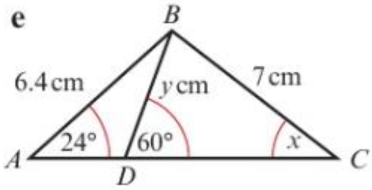


e



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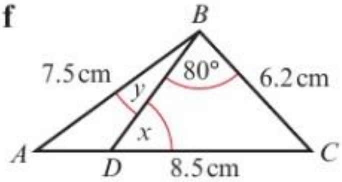


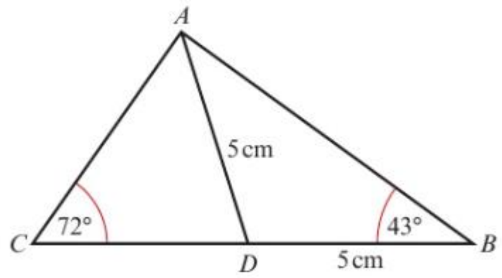
P **8** Town B is 6 km, on a bearing of 020°, from town A. Town C is located on a bearing of 055° from town A and on a bearing of 120° from town B. Work out the distance of town C from:

a town A **b** town B

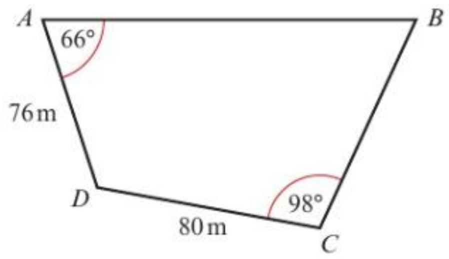
Problem-solving

Draw a sketch to show the information.

9 In the diagram $AD = DB = 5$ cm, $\angle ABC = 43°$ and $\angle ACB = 72°$.
Calculate:
a AB
b CD



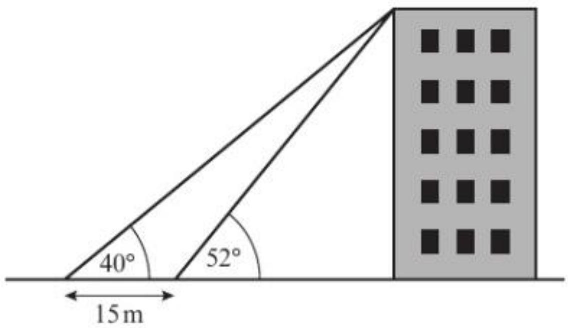
10 A zookeeper is building an enclosure for some llamas. The enclosure is in the shape of a quadrilateral as shown.
If the length of the diagonal BD is 136 m
a find the angle between the fences AB and BC
b find the length of fence AB



E/P **11** In $\triangle ABC$, $AB = x$ cm, $BC = (4 - x)$ cm, $\angle BAC = y$ and $\angle BCA = 30°$.
Given that $\sin y = \frac{1}{\sqrt{2}}$, show that
$x = 4(\sqrt{2} - 1)$ **(5 marks)**

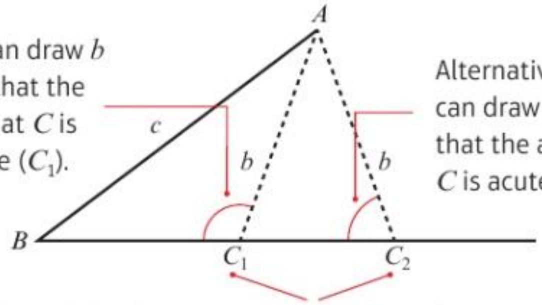
Problem-solving
You can use the value of sin y directly in your calculation. You don't need to work out the value of y.

E/P **12** A surveyor wants to determine the height of a building. She measures the angle of elevation of the top of the building at two points 15 m apart on the ground.
a Use this information to determine the height of the building. **(4 marks)**
b State one assumption made by the surveyor in using this mathematical model. **(1 mark)**



For given side lengths b and c and given angle B, you can draw the triangle in two different ways.

You can draw b such that the angle at C is obtuse (C_1).

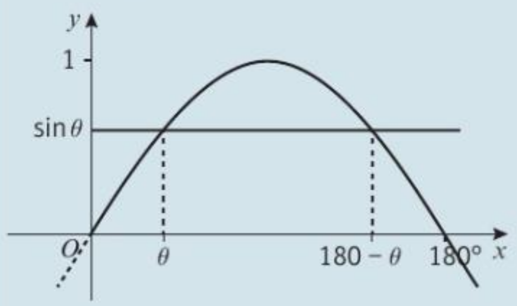


Alternatively you can draw b such that the angle at C is acute (C_2).

Since AC_1C_2 is an isosceles triangle, it follows that the angles AC_1B and AC_2B add together to make 180°.

- **The sine rule sometimes produces two possible solutions for a missing angle:**
  - **sin θ = sin (180° − θ)**

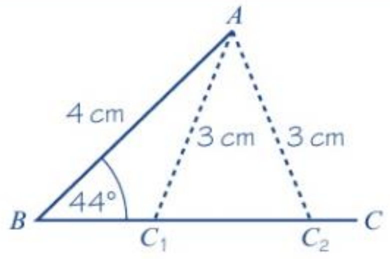
Links You can confirm this relationship by considering the graph of $y = \sin x$.



→ Section 9.5 and Chapter 10

Example 7

In $\triangle ABC$, $AB = 4$ cm, $AC = 3$ cm and $\angle ABC = 44°$. Work out the two possible values of $\angle ACB$.



$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{4} = \frac{\sin 44°}{3}$$

$$\sin C = \frac{4 \sin 44°}{3}$$

So $C = 67.851... = 67.9°$ (3 s.f.)

or $C = 180 - 67.851... = 112.14...$

$= 112°$ (3 s.f.)

Problem-solving

Think about which lengths and angles are fixed, and which ones can vary. The length AC is fixed. If you drew a circle with radius 3 cm and centre A it would intersect the horizontal side of the triangle at two points, C_1 and C_2.

Use $\frac{\sin C}{c} = \frac{\sin B}{b}$, where $b = 3$, $c = 4$, $B = 44°$.

As $\sin(180 - \theta) = \sin\theta$.

Exercise 9C

Give answers to 3 significant figures, where appropriate.

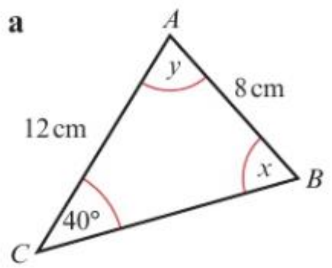
1 In $\triangle ABC$, $BC = 6$ cm, $AC = 4.5$ cm and $\angle ABC = 45°$.

a Calculate the two possible values of $\angle BAC$.

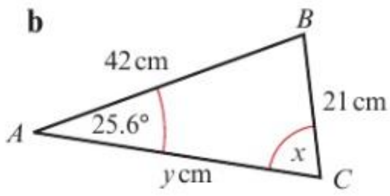
b Draw a diagram to illustrate your answers.

2 In each of the diagrams shown below, calculate the possible values of x and the corresponding values of y.

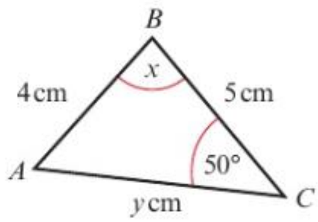
a



b



c



3 In each of the following cases $\triangle ABC$ has $\angle ABC = 30°$ and $AB = 10$ cm.

a Calculate the least possible length that AC could be.

b Given that $AC = 12$ cm, calculate $\angle ACB$.

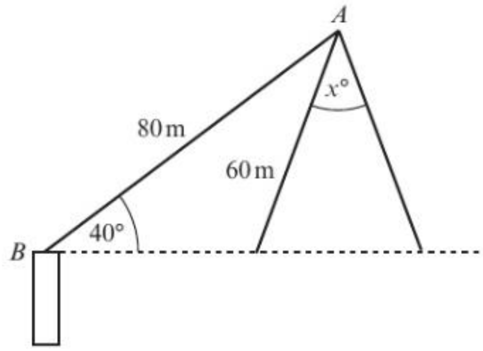
c Given instead that $AC = 7$ cm, calculate the two possible values of $\angle ACB$.

Ⓟ **4** Triangle ABC is such that $AB = 4$ cm, $BC = 6$ cm and $\angle ACB = 36°$. Show that one of the possible values of $\angle ABC$ is 25.8° (to 3 s.f.). Using this value, calculate the length of AC.

Ⓟ **5** Two triangles ABC are such that $AB = 4.5$ cm, $BC = 6.8$ cm and $\angle ACB = 30°$. Work out the value of the largest angle in each of the triangles.

E/P **6 a** A crane arm AB of length 80 m is anchored at point B at an angle of 40° to the horizontal. A wrecking ball is suspended on a cable of length 60 m from A. Find the angle x through which the wrecking ball rotates as it passes the two points level with the base of the crane arm at B. **(6 marks)**

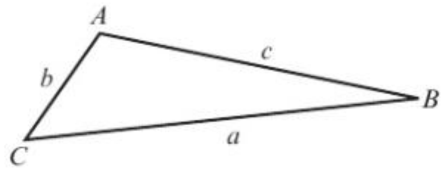
b Write down one modelling assumption you have made. **(1 mark)**



9.3 Areas of triangles

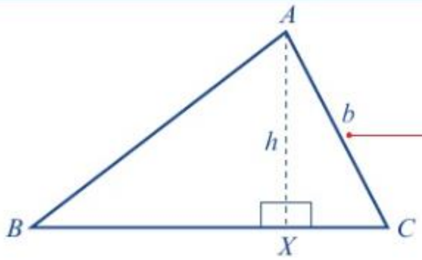
You need to be able to use the formula for finding the area of any triangle when you know two sides and the angle between them.

- **Area =** $\frac{1}{2}ab \sin C$



Hint As with the cosine rule, the letters are interchangeable. For example, if you know angle B and sides a and c, the formula becomes Area = $\frac{1}{2}ac \sin B$.

A proof of the formula:



The perpendicular from A to BC is drawn and it meets BC at X. The length of $AX = h$.

Area of $\triangle ABC = \frac{1}{2}ah$ (1)

But $h = b \sin C$ (2)

So Area $= \frac{1}{2}ab \sin C$

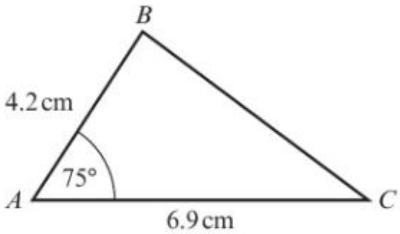
Area of triangle = $\frac{1}{2}$ base × height.

Use the sine ratio $\sin C = \frac{h}{b}$ in $\triangle AXC$.

Substitute (2) into (1).

Example 8

Work out the area of the triangle shown below.



Online Explore the area of a triangle using GeoGebra.

Area $= \frac{1}{2}bc \sin A$

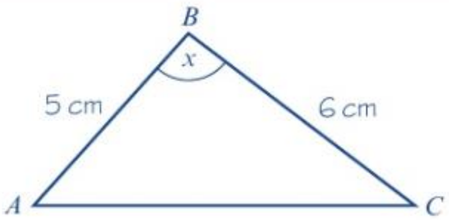
Area of $\triangle ABC = \frac{1}{2} \times 6.9 \times 4.2 \times \sin 75° \text{ cm}^2$

$= 14.0 \text{ cm}^2$ (3 s.f.)

Here $b = 6.9$ cm, $c = 4.2$ cm and angle $A = 75°$, so use:
Area $= \frac{1}{2}bc \sin A$.

Example 9

In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = x$. Given that the area of $\triangle ABC$ is 12 cm^2 and that AC is the longest side, find the value of x.



Area $= \frac{1}{2}ac \sin B$

Area $\triangle ABC = \frac{1}{2} \times 6 \times 5 \times \sin x \text{ cm}^2$

So $12 = \frac{1}{2} \times 66 \times 5 \times \sin x \text{ cm}^2$

So $\sin x = 0.8$

$x = 126.86...$

$= 127°$ (3 s.f.)

Here $a = 6$ cm, $c = 5$ cm and angle $B = x$, so use:
Area $= \frac{1}{2}ac \sin B$.

Area of $\triangle ABC$ is 12 cm^2.

$\sin x = \frac{12}{15}$

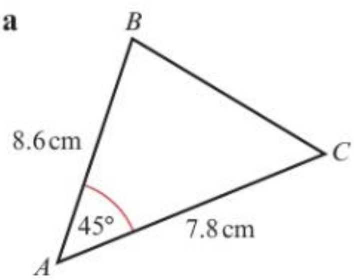
Problem-solving

There are two values of x for which $\sin x = 0.8$, $53.13...°$ and $126.86...°$, but here you know B is the largest angle because AC is the largest side.

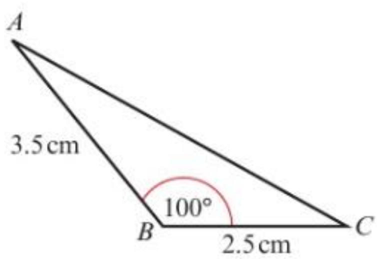
Exercise 9D

1 Calculate the area of each triangle.

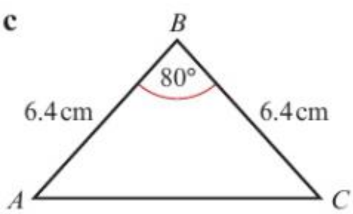
a



b

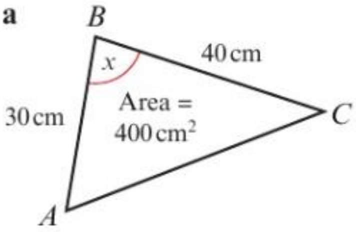


c



2 Work out the possible sizes of x in the following triangles.

a



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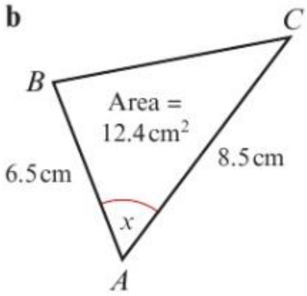
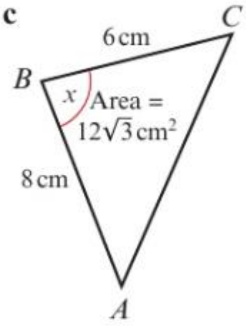
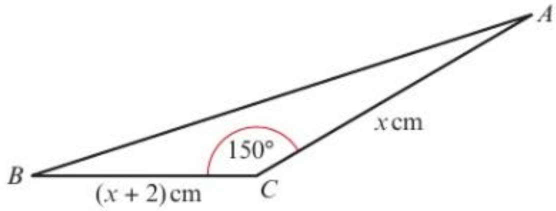


c



3 A fenced triangular plot of ground has area $1200\,\text{m}^2$. The fences along the two smaller sides are 60 m and 80 m respectively and the angle between them is θ. Show that $\theta = 150°$, and work out the total length of fencing.

(P) **4** In triangle ABC, $BC = (x + 2)$ cm, $AC = x$ cm and $\angle BCA = 150°$. Given that the area of the triangle is $5\,\text{cm}^2$, work out the value of x, giving your answer to 3 significant figures.



(E/P) **5** In $\triangle PQR$, $PQ = (x + 2)$ cm, $PR = (5 - x)$ cm and $\angle QPR = 30°$. The area of the triangle is $A\,\text{cm}^2$.

a Show that $A = \frac{1}{4}(10 + 3x - x^2)$. **(3 marks)**

b Use the method of completing the square, or otherwise, to find the maximum value of A, and give the corresponding value of x. **(4 marks)**

(E/P) **6** In $\triangle ABC$, $AB = x$ cm, $AC = (5 + x)$ cm and $\angle BAC = 150°$. Given that the area of the triangle is $3\frac{3}{4}\,\text{cm}^2$

Problem-solving

x represents a length so it must be positive.

a Show that x satisfies the equation $x^2 + 5x - 15 = 0$. **(3 marks)**

b Calculate the value of x, giving your answer to 3 significant figures. **(3 marks)**